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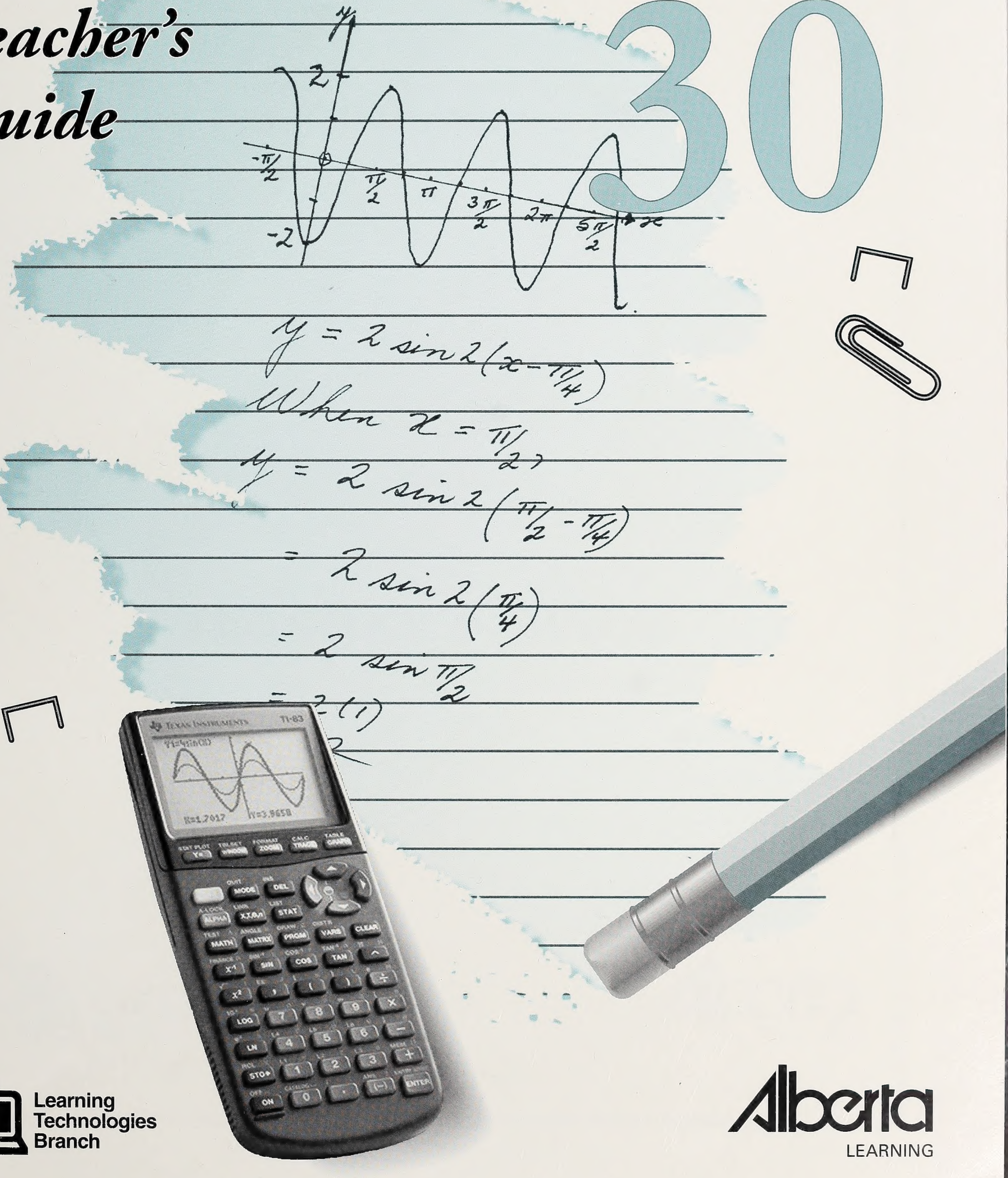
Pure MATHEMATICS 30

Teacher's Guide



Learning Technologies Branch

Alberta LEARNING

Pure Mathematics 30

TEACHER'S GUIDE

Learning
Technologies
Branch

Alberta
LEARNING

NOTE: This Pure Mathematics 30 Teacher's Guide contains the Final Test and answers to the Final Test and to the teacher-assessed assignments; therefore, teachers should at all times keep it secure against unauthorized student access. The Final Test should be accessible to students only in supervised situations.

Pure Mathematics 30
Modules 1–7
Teacher's Guide
Learning Technologies Branch
ISBN 0-7741-1633-1

This document is intended for	
Students	
Teachers	✓
Administrators	
Parents	
General Public	
Other	

You may find the following Internet sites useful:

- Alberta Learning, **http://www.learning.gov.ab.ca**
- Learning Technologies Branch, **http://www.learning.gov.ab.ca/ltb**
- Learning Resources Distributing Centre, **http://www.learning.gov.ab.ca/lrdc**

The use of the Internet is optional. Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

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Teachers

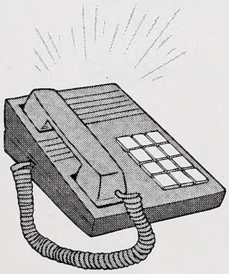
Register with the Learning Technologies Branch

The Learning Technologies Branch is dedicated to upgrading and continually improving your Teacher's Guide so that the guide accurately reflects any necessary revisions we have had to make in the Student Module Booklet, Assignment Booklet, or the sample Final Test. The types of revisions that will be made are those that make the course more accurate, more current, or more effective.

LTB will send you the **latest enhancements** or **minor upgrades** for your Teacher's Guide if you register with us. To do so, please fill out the registration portion of this page and send this page to: Learning Technologies Branch, Box 4000, Barrhead, Alberta T7N 1P4, Attention: Instructional Design and Development.

LTB Teacher's Guide Registration Form

First Name	Surname
School Name	School Phone Number
School Address	
City	Postal Code
Course Title	Approximate Date of Purchase



You can help ensure that distance learning courseware is of top quality by letting us know of areas that need to be adjusted. Call the Learning Technologies Branch—free of charge by dialling first 310-0000, then 674-5350—and ask for the Editing Unit. Also, a teacher questionnaire has been included at the back of most Teacher's Guides. Please take a moment to fill it out and send it in.

We look forward to hearing from you!

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Introduction

A survey of these course materials will confirm that this learning package has been specially designed for many kinds of teachers working in a variety of situations.

In Which Category Do You Fit?

☐ Small Schools Teacher

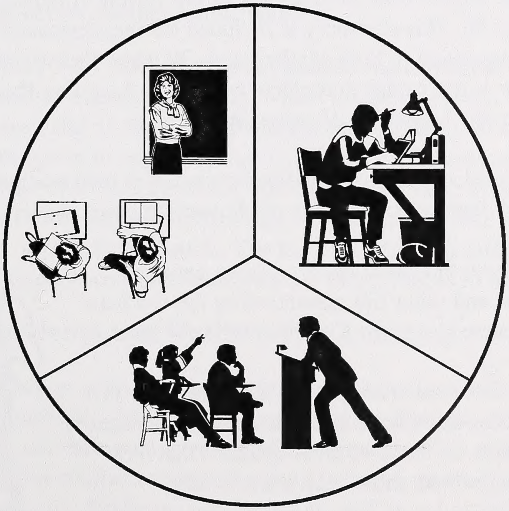
  ☐ inexperienced
  ☐ experienced, but in other subject areas
  ☐ experienced in teaching Pure Mathematics 30, but wanting to try a different approach

☐ Distance Learning Teacher

  ☐ travelling to schools within the region
  ☐ using facsimile and teleconferences to teach students within the region

☐ Large Schools Teacher

  ☐ inexperienced
  ☐ experienced in teaching Pure Mathematics 30, but wanting to try a different approach



Because these materials have been created by experienced classroom teachers and distance learning specialists, they have many advantages for students and teachers, regardless of the situation.

Advantages for Students

Materials

- incorporate a strong learner-centred philosophy
- promote such qualities in the learner as autonomy, independence, and flexibility
- are developed through media that suit the needs and circumstances of the learner
- reflect the experiential background of Alberta students
- provide opportunities by overcoming barriers that result from geographical location
- promote individualized learning, allowing learners to work at their own pace

Advantages for Teachers

Materials

- allow teachers maximum teaching time and minimize preparation time
- include different routes through the materials to suit different learners
- incorporate a wide range of teaching strategies, in particular those using independent and individual learning
- deliver curriculum designed by education specialists that reflects the Alberta Learning Program of Studies with an emphasis on Canadian content
- provide learning materials that are upwardly compatible with advanced educational technology

Does this learning package sound like something you could use?

This Teacher's Guide begins with an overview of the current Alberta Learning Program of Studies for Pure Mathematics 30. This summary is included for inexperienced teachers or those teachers who have found themselves teaching Pure Mathematics 30 when their training is in other subject areas. This brief description is not meant to replace the Alberta Learning Program of Studies, but rather to help teachers confirm the highlights of the program.

Other parts of this introduction have also been included to help teachers become familiar with this learning package and determine how they might want to use it in their classrooms.

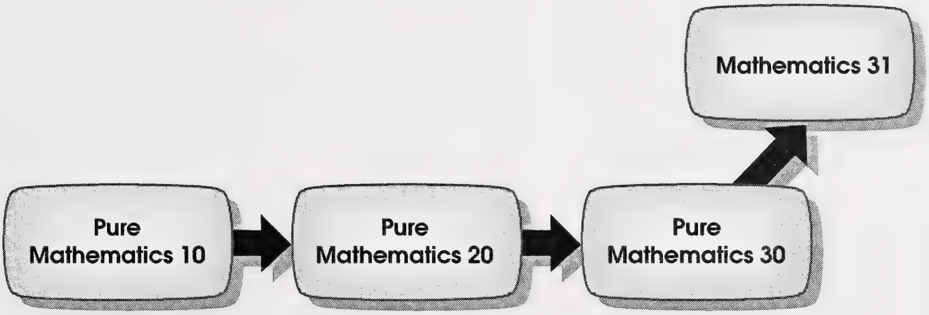
Beyond the introduction, the guide itself contains answers, models, explanations, and other tips generated by the teachers who authored this course.

The Student Module Booklets, Assignment Booklets, and TGs are the products of experienced classroom teachers and distance learning specialists. It is the hope of these teachers that their experience can be shared with those who want to take advantage of it.

Overview of the Program of Studies

The distance learning materials for Pure Mathematics 30 reflect the beliefs, goals, and expected student outcomes of the document *The Common Curriculum Framework for K–12 Mathematics Grade 10 to Grade 12: Western Canadian Protocol for Collaboration in Basic Education,* 1996. This document provides the basis for the programs of study in Alberta, British Columbia, Manitoba, Saskatchewan, the Northwest Territories, and the Yukon. The results are two programs of study that include ten courses: Applied Mathematics 10–20–30, Applied Mathematics 10b–20b, Pure Mathematics 10–20–30, and Pure Mathematics 10b–20b. The Pure Mathematics 10–20–30 program is made up of outcomes, some of which are common to both the pure and applied programs.

The prerequisite for Pure Mathematics 30 is achieving the acceptable standard for Pure Mathematics 20 or Applied Mathematics 20 and Pure Mathematics 20b combined. The courses in the Pure Mathematics 10–20–30 sequence are shown in the following diagram.



For Mathematics 31, which is not part of the Western Canadian Protocol course structure, the prerequisite or corequisite is Pure Mathematics 30. It is highly recommended that students complete Pure Mathematics 30 before starting Mathematics 31.

The Pure Mathematics 10–20–30 sequence emphasizes mathematical theory and the testing of hypotheses. The pure mathematics approach, which is often symbolic and deductive, endeavours to show that concepts are valid all the time or within a well-defined set of restrictions. Real-life examples are then presented to apply previously learned mathematical concepts and procedures. Algebra is taught on a "when-needed basis" and for its own sake. Students are required to demonstrate effective communication skills and are expected to take responsibility for their learning. Technology is a part of pure mathematics. The use of a graphing calculator is the primary tool; however, the use of spreadsheets and databases can be profitable in many contexts.

Beliefs About Students and Mathematics Learning

Students are active learners who have individual interests and abilities and needs; they hold a range of attitudes about mathematics and life. Students learn by attaching meaning to what they do; they must be able to construct their own meaning of mathematics. This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. Students need numerous and varied experiences in order to appreciate the usefulness of mathematics and to view mathematics as an integrated whole.

Goals for Students

Mathematics education should prepare students to

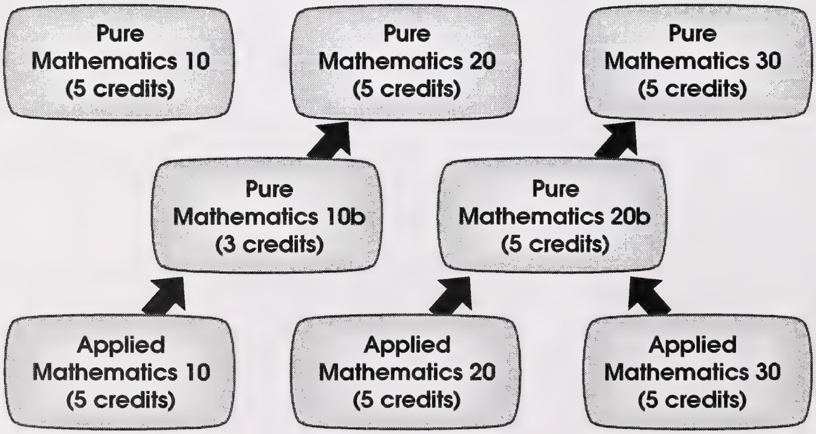
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- commit themselves to lifelong learning
- become mathematically literate adults using mathematics to contribute to society

Expected Student Outcomes

The content of the Common Curriculum Framework is stated in terms of outcomes. These outcomes are measurable and identify what the students are expected to do. The outcomes are organized within four strands: Number, Patterns and Relations, Shape and Space, and Statistics and Probability. There are eight mathematical processes students must encounter in a mathematics program in order to achieve these outcomes. These processes include communication, connection, estimation, mental mathematics, problem solving, reasoning, technology, and visualization. The distance learning materials incorporate these eight interrelated mathematical processes that are intended to permeate teaching and learning.

Transferring from Applied Mathematics to Pure Mathematics

Students, for a variety of reasons, may want to transfer from the applied mathematics course sequence to the pure mathematics course sequence. The outcomes that are common to pure mathematics and applied mathematics need not be repeated. For those students who want to transfer after meeting the acceptable standard in Applied Mathematics 10, the prerequisite course is Pure Mathematics 10b, a three-credit course. However, students may opt to complete the entire Pure Mathematics 10 course and earn five credits. For those students who want to transfer after meeting the acceptable standard in Applied Mathematics 20, the prerequisite course is Pure Mathematics 20b, a five-credit course. The two bridging courses are shown in the following diagram.


Pure Mathematics 10 (5 credits)
Pure Mathematics 20 (5 credits)
Pure Mathematics 30 (5 credits)
Pure Mathematics 10b (3 credits)
Pure Mathematics 20b (5 credits)
Applied Mathematics 10 (5 credits)
Applied Mathematics 20 (5 credits)
Applied Mathematics 30 (5 credits)

Overview of Pure Mathematics 30

Pure Mathematics 30 contains seven modules. Because of the sequential nature of the course, the modules should be approached in order. The percentage appearing in each box suggests the approximate proportion of work time devoted to that module. The distance learning materials have been designed to indicate clearly the outcomes specified in the program of studies.

The Pure Mathematics 30 course emphasizes mathematical theory and the testing of hypotheses. The approach to mathematics is deductive and symbolic. Real-life problems are introduced to demonstrate the applications of theory and procedures. Students must communicate effectively. Students are expected to explain, to illustrate, to reason, and to make connections. Multiple solution strategies to problems and problem contexts will be expected as students work through both routine and non-routine problems.

Every attempt has been made to give students a variety of activities and to make mathematics real and interesting. The graphing calculator, the computer, and the Internet are integrated to cater to learning styles of individual students. Therefore, this distance learning course is more than a workbook; it is a complete learning package.

Pure Mathematics 30

Module 1
Transformations
15%

Module 2
Exponents and Logarithms
10%

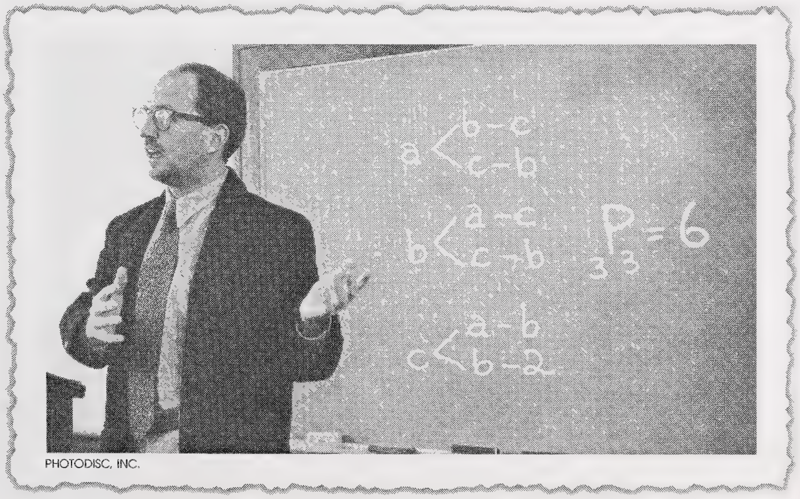
Module 3
Sequences and Series
10%

Module 4
Conics
15%

Module 5
Trigonometry
15%

Module 6
Combinatorics
15%

Module 7
Probability and Statistics
20%


$a<\begin{matrix}b-c\\c-b\end{matrix}$
$b<\begin{matrix}a-c\\c-b\end{matrix}$
$c<\begin{matrix}a-b\\b-a\end{matrix}$
$_3P_3=6$

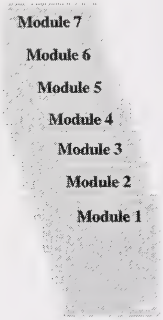
PHOTODISC, INC.

Structure of the Learning Package

Basic Design

This learning package involves many other components in addition to the Teacher's Guide. A survey of the components will show you that the course is divided into clumps of learning called modules. For each module there are two print components: a Student Module Booklet and the Assignment Booklets.

Student Module Booklets



Student Module Booklets contain guided activities that instruct students in a relevant, realistic setting.

These booklets have been specially designed to promote such qualities in the learner as autonomy, independence, and flexibility. Writers have incorporated such teaching strategies as working from the concrete to the abstract, linking the old to the new, getting students actively involved, and using advance, intermediate, and post organizers. Many other techniques are used to cater to individual learning styles and preferences. The materials have been designed to include a variety of pathways and options because they are intended for a broad range of use within and beyond Alberta.

Contents
Overview Evaluation
Section 1 Activity 1 Activity 2 etc.
Section 2 Activity 1 Activity 2 etc.
Module Summary
Appendix

The structure of the Student Module Booklets follows a systematic design. Each booklet begins with a detailed table of contents that shows the students all of the main steps; this page acts as an organizer for students. The Module Overview introduces the module topic or theme and includes a graphic representation to help visual learners and poor readers. This introduction also includes an evaluation statement, so students are informed of the weightings of each assignment.

The body of the Student Module Booklet is made up of two or more closely related sections. Each section contains student activities that develop skills and knowledge centred around a theme. The activities may include print, audio, video, computer, or Internet involvement. Computer and Internet activities are optional. At times, the student and the teacher are allowed to choose the activity that best suits the student's needs and interests. Each section also includes other activities such as the Extra Help and Enrichment as optional pathways. This flexibility caters to each student's personal situation.

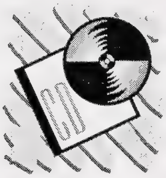
Following the last section is a modular summary that focuses on the skills and strategies that the student has learned. The Student Module Booklet ends with an Appendix that includes a Glossary and Suggested Answers for the self-assessment work.

Assignment Booklets

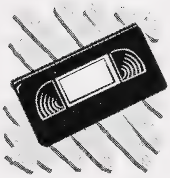
Accompanying each Student Module Booklet are the Assignment Booklets. The activities in the Assignment Booklets can be used for both formative and summative assessments. Students should complete these assignments when they have thoroughly reviewed the other module materials. Students should submit their work for evaluation as soon as they have completed each Assignment Booklet. The Assignment Booklets have been designed for both in-classroom use and for distance learning.

Media

INTERNET



CD-ROM



VIDEOCASSETTE

Internet references are found throughout the course; exploring those references, however, is optional for students. In most cases, students are given suggested topics that they may explore through the Internet's search engines. In rare cases, students might be given a specific uniform resource locator (URL), but they should be aware that these addresses are subject to change.

Updated information about the Learning Technologies Branch and this course and others can be found by starting at the Learning Technologies Branch's home page at

http://www.learning.gov.ab.ca/ltb/

or by going straight to the LTB Courses page at

http://www.learning.gov.ab.ca/ltb/400/courses.html

There are also activities in this course that direct students to view videos. The use of these videos is optional. A list of these optional videos appears on the next page. If you want students to have access to these videos, arrangements should be made to ensure they are available when students need them. More information about the videos appears later in this manual.

Textbooks and Reference Books

MATHPOWER™ 12, Western Edition, McGraw-Hill Ryerson (2000)

Materials, Media, and Equipment

Mandatory Components

Equipment (Hardware)	Media	Materials
• graphing calculator • computer	• spreadsheet software (such as *Claris Works*™ and *Excel*®) • Pure Mathematics 30 Companion CD	• Teacher's Guide for Pure Mathematics 30 • one complete set of Student Module Booklets (7) and Assignment Booklets (14) for each student • the *MATHPOWER*™ *12*, Western Edition, textbook, published by McGraw-Hill Ryerson (2000) • There is a Final Test.

Videocassettes or laser videodiscs used in the course may be available from the Learning Resources Distributing Centre or ACCESS. You may also wish to call your regional library service for more information.

Optional Components

Equipment (Hardware)	Media	Materials
• videocassette player	• the Internet • optional videos – *Math Factor Review* – Educational Video Resources, *The TI-83 Graphing Calculator Video Tutor*	

Using This Learning Package in the Classroom

Conventional Classroom

Whether your classroom has desks in rows or tables in small groups, you may be most comfortable with a learning system that you can use with all your students in a paced style. In other words, you may want a package that will suit all of your students, so they can move through the materials as one group or several small groups. Because these materials contain different routes or pathways within each module, they can address various learning styles and preferences. The materials also include many choices within the activities to cater to different thinking levels and ability levels. Because of their versatility and flexibility, these materials can easily suit a conventional classroom.

Open Learning Classroom

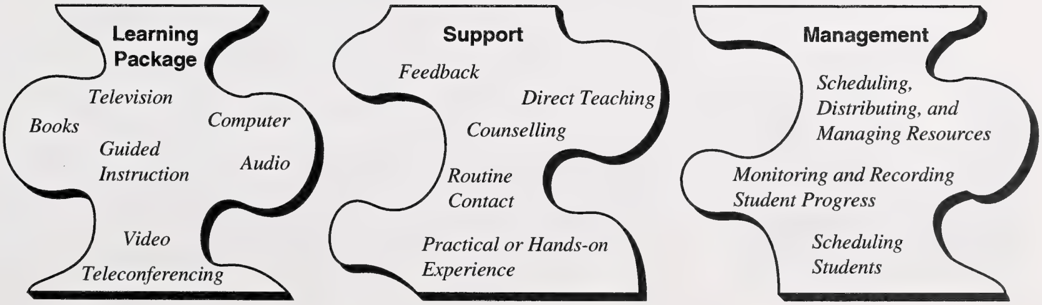
Open learning is the concept of opening up opportunities by overcoming barriers of time, pace, and place by giving the learners a package specially designed to enable them to learn on their own for at least some of the time.

Such a concept is not new. Many teachers can recite attempts to establish an individualized learning system as they recognized the importance of trying to personalize courseware to meet each individual's needs. But these past efforts often failed because of lack of time and lack of quality materials that conformed to Alberta specifications.

Owing to advanced educational technology and improved Alberta-specific learning packages, a student-centred approach is now possible. Improved technology now allows us to provide support to learners individually, regardless of their pace or location. A teacher cannot be in twenty-eight places at one time offering guidance. Media and a well-designed learning package, however, can satisfy individual needs. Technology can also help provide an effective management system needed to track the students as they progress independently through the materials.

The key to a successful open learning system depends on three vital elements: a learning package specially designed to enable students to learn effectively on their own for at least some of the time; various kinds of learner support; and a management system and style that ensures that the open learning system runs smoothly.

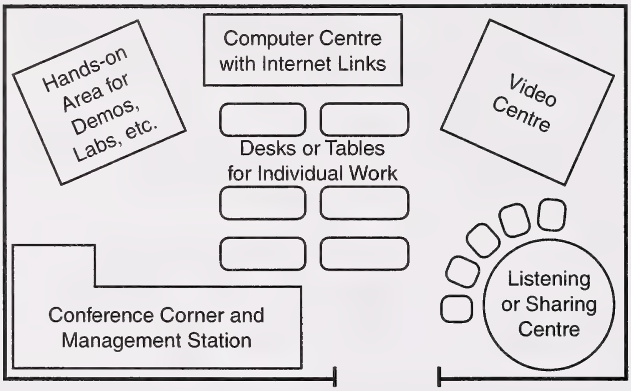
The Key to a Successful Open Learning System



Learning Package

The specially designed learning package needed for a successful open learning system has been developed for you. The objectives teach current Alberta specifications using strategies designed for individualized instruction. As the teacher, you need to be sure to have all of the components in the learning package available to students as needed.

If you are able to acquire media and appropriate hardware to meet your class needs, media centres can be established.


Hands-on Area for Demos, Labs, etc.
Computer Centre with Internet Links
Video Centre
Desks or Tables for Individual Work
Conference Corner and Management Station
Listening or Sharing Centre

You may not have the luxury to have enough hardware to set up a permanent video or computer centre in your classroom. In that case, students should be encouraged to plan ahead. Perhaps every three to five days they should preview their materials and project when they would need a certain piece of media. This would allow you to group students, if necessary, or reserve media as required.

Support

Support is definitely a key element for successful learning, and when you're planning an individualized, non-paced program, you need to carefully plan when and how support will be given.

The materials contain a form of consistent support by providing immediate feedback for activities included in the Student Module Booklet. Students have solutions, models, explanations, and guides included in the Appendix of every booklet. These aids are included so students can receive immediate feedback to clarify and reinforce their basic understanding before they move on to higher levels of thinking.

As the teacher, you may need to offer more support and personal guidance to those students having difficulty. The activities include choices and pathways. If a student is struggling, you may need to encourage that student to work on all of the choices rather than on only one. This would provide additional instruction and practice in a variety of ways.

You may also have to reinforce the need for students to do the sectional activities carefully and thoroughly before attempting the assignments in the Assignment Booklets.

Another form of support is routine contact with each individual. This contact might be achieved with a biweekly conference scheduled by you; or, as students reach a certain point (e.g., after each section is completed), they may be directed to come to the conference area.

Special counselling may be needed to help students through difficult stages. Praise and encouragement are important motivators, particularly for those students who are not used to working independently.

Direct teaching may be needed and scheduled at certain points in the program. This teaching might involve small groups or a large group. It might be used to take advantage of something timely (e.g., election, eclipse), something prescheduled like the demonstration of a process, or something involving students in a hands-on, practical experience.

Support at a distance might include tutoring by phone, teleconferencing, faxing, or planned visits. These contacts are the lifeline between learners and distance education teachers, so a warm dialogue is essential.

Management

Good management of an open learning system is essential to the success of the program. Leadership and promotion of the system are essential. The following areas also need action to ensure that the system runs smoothly:

- Scheduling, Distributing, and Managing Resources—As discussed earlier, in order to do these tasks efficiently you may need to develop media centres or a system for students to reserve the necessary resources.
- Scheduling Students—Students and teachers should work together to establish goals, course completion timelines, and daily timelines. Although students may want to study for long periods of time (e.g., all morning), teachers should discourage this practice. Concentration, retention, and motivation are improved by taking scheduled breaks.
- Monitoring Student Progress—You will need to record the date that each student completes each module and the Final Test. Your data might also include the projected date of completion if you are using a student-contract approach.

Sample of a Student Progress Chart

Pure Mathematics 30		Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Module 7	Final Test
Billy Adams	P								
	A								
Louise Despins	P								
	A								
Violet Klaissian	P								
	A								
P = Projected Completion Date A = Actual Completion Date									

The student could keep a personal log as well. Such tracking of data could be stored easily on a computer.

- Recording Student Assessments—You will need to record the marks awarded for each module based on the work completed in the Assignment Booklets. The marks from these Assignment Booklets will contribute to a portion of the student's final mark. Other criteria may also be added (a special project, effort, attitude, etc.). Whatever the criteria, they should be made clear to all students at the beginning.

Sample of a Student Assessment Chart

Pure Mathematics 30	Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Module 7	Year's Average	Other	Final Test	Final Mark
Billy Adams											
Louise Despins											
Violet Klaissian											

Letter grading could easily be substituted.

- Recording Effectiveness of System—Keep ongoing records of how the system is working. This data will help you in future planning.

Sample of a System Assessment Chart

Pure Mathematics 30: Module 1			
Date	Student Module Booklet	Assignment Booklet	Resources/Media

The Role of the Teacher in an Open Learning Classroom

The teachers in a conventional classroom spend a lot of time talking to large groups of learners. The situation in open learning requires a different emphasis. Teachers will probably meet learners individually or in very small groups.

With this approach, it is necessary to move beyond the idea of a passive learner depending largely on a continually supportive teacher. The teacher must aim to build the student's confidence, to stimulate the learner into self-reliance, and to guide the learner to take advantage of routes that are most meaningful and applicable to the learner.

These materials are student centred, not teacher centred. The teacher needs to facilitate learning by providing general support to the learner.

Evaluation

Evaluation is important to the development of every learner. In order to identify strengths and weaknesses, both students and teachers need to gather and analyse information and make decisions.

These specially designed learning packages contain many kinds of informal and formal evaluation.

Informal Evaluation

Observation

In the classroom, the teacher has the opportunity to see each student perform every day and to become aware of the level and nature of each student's performance.

Observations are more useful if they are recorded in an organized system. The following list of questions is a sample of types of observations and how they can be collected.

Observation Checklist

	B. Adams	L. Despins	V. Klaissian	H. Smith	K. Dalley
1. Does the student approach the work in a positive manner?					
2. Is the student struggling with the reading level?					
3. Does the student make good use of time?					
4. Does the student apply an appropriate study method?					
5. Can the student use references effectively?					

Observation may suggest a need for an individual interview with a student.

Individual Conferences

Individual conferences may be paced (scheduled) by the calendar, designated at certain points in the module, or they may be set up only as needed or requested.

During these conferences, the teacher can determine the student's progress and can assess the student's attitudes towards the subject, program, school, and self, as well as the student's relationship with other students. With guided questions, the teacher can encourage oral self-assessment; the student can discuss personal strengths or weaknesses in regard to the particular section, module, or subject area.

Self-Appraisal

Self-appraisal helps students recognize their own strengths and weaknesses. Through activities that require self-assessment, students also gain immediate feedback and clarification at early stages in the learning process. Teachers need to promote a responsible attitude towards these self-assessment activities. Becoming effective self-assessors is a crucial part of becoming autonomous learners. By instructing, motivating, providing positive reinforcement, and systematically supervising, teachers will help students develop a positive attitude towards their own progress.

For variation, students may be paired and peer-assessing may become part of the system. The teacher may decide to have the student self-assess some of the activities, to have a peer assess other activities, and to become directly involved in assessing the remainder of the activities.

When the modular activities have been assessed, students should be directed to make corrections. This task should be made clear to students right from the beginning. It is important to establish the correct association between the question and the response to clarify understanding, aid retention, and be of use for study purposes.

Many of the activities include choices for students. If students are having difficulty, more practice may be warranted, and students may need to be encouraged to select more of the choices.

Each section within a Student Module Booklet includes additional types of activities called Extra Help and Enrichment. Students are expected to be involved in the decision as to which pathway best suits their needs. They may decide to do both.

Self-appraisal techniques can also be introduced at the individual conferences. Such questions as the following might be included:

- What steps are you taking to improve your understanding of this topic?
- What method of study do you use most?
- How do you organize the material to help you remember it?
- What steps do you follow when you complete an assignment?
- What could you do to become an even better reader?
- Do you have trouble following directions?
- Did you enjoy this module?

A chart or checklist could be used for recording responses.

Assignments

Each module contains separate booklets called Assignment Booklets. These booklets are informal evaluation tools for assessing the knowledge or skills that the student has gained from the module. **The student's mark for the module may be based solely on the outcome of learning evident in the Assignment Booklets; however, you may decide to establish a value for other variables such as attitude or effort.** It is important that you establish at the beginning of the course and each module those outcomes that will be evaluated and ensure that all students clearly understand what is expected.

Formal Evaluation

This Teacher's Guide includes a formal Final Test that can be photocopied for each member of the class. The test, closely linked to the learning outcomes stated in the Student Module Booklets, gives the teacher precise information concerning what each student can or cannot do. Answers, explanations, and marking guides are also included. The value of the Final Test and each module is the decision of the classroom teacher. Following is a suggestion only.

Module 1	**7%**
Module 2	**5%**
Module 3	**5%**
Module 4	**6%**
Module 5	**7%**
Module 6	**7%**
Module 7	**13%**
Final Test	**50%**

Remember: Pure Mathematics 30 is a diploma examination subject. A student's final course mark is a blend of the teacher-assigned mark and the diploma examination mark.

Introducing Students to the System

Your initiation to these learning materials began with a basic survey of what was included and how the components varied. This same process should be repeated with the class. After the materials have been explored, a discussion might include the advantages and the disadvantages of learning independently or in small groups. The roles of the student and teacher should be analysed. The necessary progress checks and rules need to be addressed. Your introduction should motivate students and build a responsible attitude towards learning autonomously.

Skill Level

It is important for students to understand that there are certain skills that they will need in order to deal successfully with the course materials. Those skills are as follows:

- understanding and using instructional materials (table of contents, index, list of illustrations, appendices, bibliography, and glossary)
- interpreting maps, graphs, and charts
- using reference materials
- recognizing special symbols
- using a graphing calculator
- using a computer and spreadsheet and database software
- understanding and using different forms of media

Other general skills follow: using reliable study methods, outlining, and learning to read at a flexible rate.

To decide the level and amount of instruction needed to accommodate the varied levels among students, you may wish to prepare and administer skill inventories or pretests. If most students need help with a particular skill, you may want to plan a total-class instructional session. If only certain students lack a skill, you may want to set up a temporary skill group to help students who need it, or you may want to develop a skills file for this purpose.

Reading Level

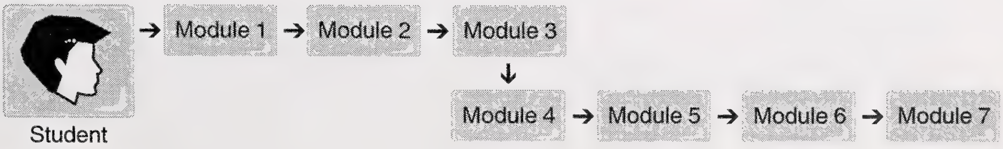
These course materials are largely print based, but poorer readers need not be discouraged. It is important that you assure students that these materials have been designed for easy reading. The authors have employed special strategies to reduce and control the reading level. Some of them are as follows:

- the conscious selection of vocabulary and careful structuring of sentences to keep the materials at an independent reading level

- the integration of activities, examples, and illustrations to break text into appropriate-sized chunks
- the inclusion of many kinds of organizers (advance, graphic, intermediate, concept mapping, post organizers) to help give students a structure for incorporating new concepts
- the recognition that vocabulary and concepts are basic to understanding the content and, thus, must be handled systematically (defined in context, signalled in marginal notes or footnotes, and cited in a glossary)
- the acknowledgement that background knowledge and experience play a vital role in comprehension
- the systematic inclusion of illustrations, videos, audiocassettes, and software to help poorer readers and visual learners
- a use of a variety of formats (paragraphs, lists, charts, etc.) to help struggling readers who do not absorb or retain main ideas easily in paragraph format
- the inclusion of media and activity choices to encourage an active rather than passive approach
- the development of instruction in a meaningful setting rather than in a contrived, workbook style
- the use of purposeful reading, viewing, and doing to produce better interpretation of the course materials
- the recognition that students need structured learning experiences when reading, viewing, or listening to instructional materials; and the acknowledgement that the recognized reading process provides such a structured framework by emphasizing the following phases: developing pupil readiness, determining the purpose, providing guided instruction and feedback, ensuring opportunity to reread or review if necessary, and including enrichment or extensions

To help make the learning package more readable, you can begin your course or module preparation by reading (viewing, listening to) all the related materials that are going to be used. You will need this solid understanding in order to link the content to the students' experiential base. This linking may be done through class brainstorming sessions concerning the topic or by using visuals and guided questions to predict what the course or modules might be about. Such a strategy helps poorer readers strengthen their abilities to be able to predict new vocabulary.

It is recommended that you have students begin with Module 1, because this module includes basic introductory information, and it is also recommended that you end with Module 7, because this module acts as a summary or culmination.



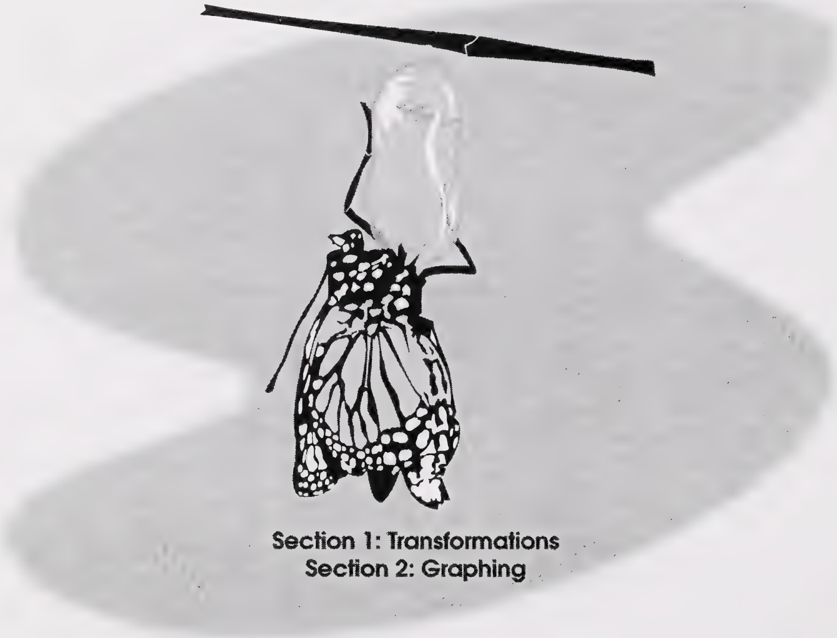
Module 1: Transformations

Overview

Section 1 of this module introduces students to the transformations of functions as tools to analyse graphs or to obtain complex graphs from basic graphs. In particular, students explore vertical and horizontal translations; reflections in the x-axis, in the y-axis, and in the line $y = x$; and both vertical and horizontal expansions and compressions. By the end of this section, students should be able to identify and describe the transformations functions have undergone from given graphs and equations. They should also be able to graph or determine the equations of a variety of functions given these transformations.

Section 2 introduces students to combinations of the transformations studied in Section 1, to the conventions used in applying several different transformations to the same function, and to additional general graphing strategies. Students then focus on reciprocal functions and absolute value functions and the strategies involved when sketching these curves.

Module 1: Transformations



Section 1: Transformations
Section 2: Graphing

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 1A	
Section 1 Assignment	**40 marks**
Assignment Booklet 1B	
Section 2 Assignment	**40 marks**
Final Module Assignment	**20 marks**
TOTAL	**100 marks**

Section 1: Transformations

Key Concepts

- transformation
- reflections in both axes and in $y = x$
- stretches of functions
- horizontal and vertical translations
- inverse
- compressions and expansions (dilations)

The basic goals of this section are to ensure that students are able to

- describe how various translations of functions affect graphs and their related equations: $y = f(x-h)$ and $y = f(x)+k$
- describe how various reflections of functions in both axes and in the line $y = x$ affect graphs and their related equations: $y = f(-x)$, $y = -f(x)$, and $y = f^{-1}(x)$
- describe how various stretches of functions (expansions and compressions) affect graphs and their related equations: $y = af(x)$ and $y = f(kx)$

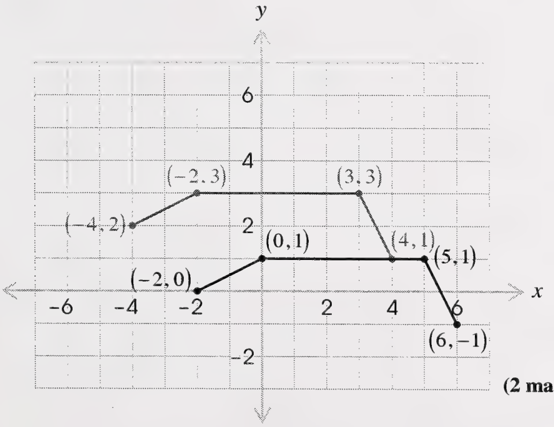
Section 1: Assignment Answer Key (40 marks)

1. Because $y-2 = f(x)$ is the same as $y = f(x)+2$, the graph of this function is the graph of $y = f(x)$ translated 2 units up. **(1 mark)**

2. The graph of $y = f(x+4)$ is the graph of $y = f(x)$ translated 4 units to the left. **(1 mark)**

3. Because the translated graph is the graph of $y = \sqrt{x}$ translated 3 units to the right and 2 units down, the equation of the translated graph is $y = \sqrt{x-3} - 2$. **(2 marks)**

4. **a.**



(2 marks)

b. Because the graph of $y+2 = f(x-2)$ is the same as $y = f(x-2)-2$, the graph of this function is the graph of $y = f(x)$ translated 2 units to the right and 2 units down. **(1 mark)**

5. Let h be the horizontal translation.

Therefore, $y=(x-h)^3+2$ is the transformed function. Because the graph of the transformed function passes through $(3,10)$, substitute $x=3$ and $y=10$ into the equation to find the horizontal translation.

$$\begin{aligned} y &= (x-h)^3+2 \\ 10 &= (3-h)^3+2 \\ 8 &= (3-h)^3 \\ \sqrt[3]{8} &= 3-h \\ 2 &= 3-h \\ h &= 3-2 \\ &= 1 \end{aligned}$$

Therefore, the new equation of the graph is $y=(x-1)^3+2$. **(3 marks)**

6. **a.** Because $y=f(-x)$ is the reflection of $y=f(x)$ in the y-axis, $y=(-x-2)^2+1$ is the reflection of $y=(x-2)^2+1$ in the y-axis. **(1 mark)**

b. The reflected graph is also the graph of $y=(x-2)^2+1$ translated 4 units to the left. **(1 mark)**

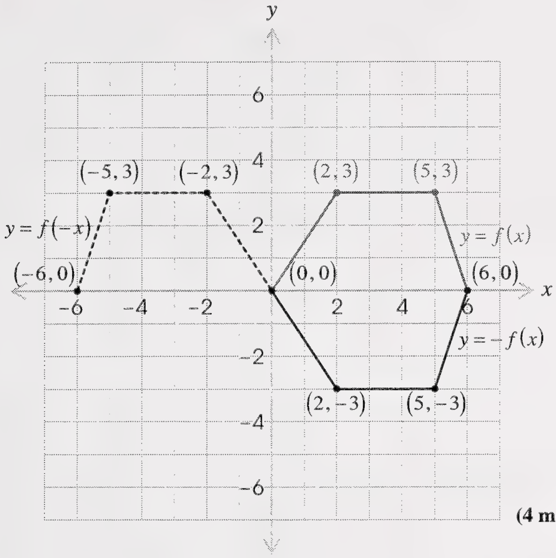
c. Show that $y=f(-x)$ is the same as $y=f(x+4)$.

LS	RS
$f(-x)$	$f(x+4)$
$=(-x-2)^2+1$	$=[(x+4)-2]^2+1$
$=[-1(x+2)]^2+1$	$=(x+2)^2+1$
$=(-1)^2(x+2)^2+1$	
$=(x+2)^2+1$	

LS = RS

(3 marks)

7.



(4 marks)

8. **a.** Let $y = x^2 - 3x$ be function *f*.

$\therefore f(x) = x^2 - 3x$

Rewrite $y = 3x - x^2$.

$$\begin{aligned} y &= 3x - x^2 \\ &= -\left(-3x + x^2\right) \\ &= -\left(x^2 - 3x\right) \\ &= -f(x) \end{aligned}$$

Therefore, $y = x^2 - 3x$ and $y = 3x^2 - x^2$ are mirror images in the *x*-axis. **(2 marks)**

b. Because $y = x^2 - 3x$ and $y = 3x - x^2$ are mirror images in the *x*-axis, they have the same *x*-intercepts.

To find the *x*-intercepts, let $y = 0$.

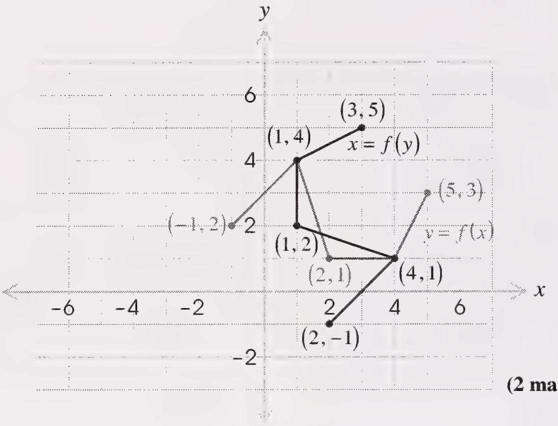
$$\begin{aligned} \therefore y &= x^2 - 3x \\ 0 &= x^2 - 3x \\ 0 &= x(x - 3) \end{aligned}$$

$\therefore x = 0$ or $x - 3 = 0$

$x = 3$

The two graphs have *x*-intercepts at $(0, 0)$ and $(3, 0)$. **(2 marks)**

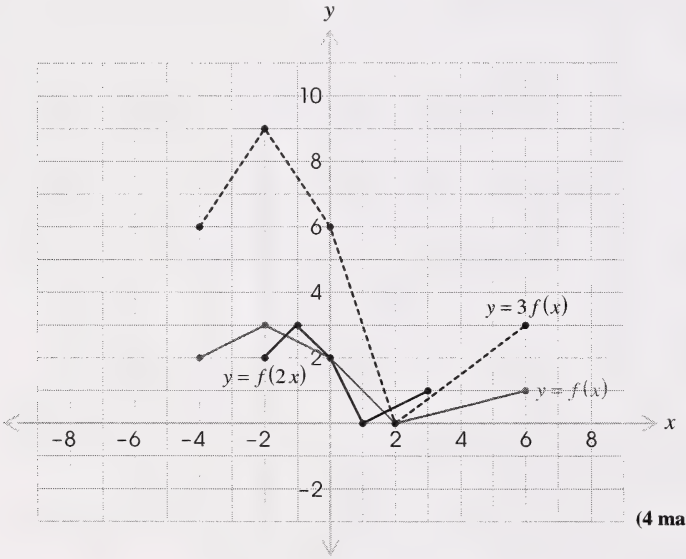
9. **a.**



(2 marks)

b. The graphs of $y = f(x)$ and $x = f(y)$ are mirror images in the line $y = x$. **(1 mark)**

10.



(4 marks)

11. **a.** The function $f(x)=\frac{2x-1}{x+2}$ is the same as $y=\frac{2x-1}{x+2}$. The inverse of *f* is $x=\frac{2y-1}{y+2}$. Solve the inverse for *y* in terms of *x*.

$$\begin{aligned} x&=\frac{2y-1}{y+2} \\ x(y+2)&=2y-1 \\ xy+2x&=2y-1 \\ xy-2y&=-2x-1 \\ y(x-2)&=-2x-1 \\ y&=\frac{-2x-1}{x-2} \end{aligned}$$

$\therefore f^{-1}(x)=\frac{-2x-1}{x-2}$ **(2 marks)**

b. Because $y=f(x)$ and $y=f^{-1}(x)$ are inverses, common points lie on the line $y=x$. **(1 mark)**

12. The graph of $y=2f(4x)$ is the graph of $y=f(x)$ expanded vertically by a factor of 2 and compressed horizontally by a factor of $\frac{1}{4}$. **(2 marks)**

13. The graph has been stretched vertically by a factor of 3.

$$\begin{aligned} \therefore\ y&=3f(x) \\ &=3\left(\frac{-2}{x^2+1}\right) \\ &=\frac{-6}{x^2+1} \end{aligned}$$

The equation of the second graph is $y=\frac{-6}{x^2+1}$. **(2 marks)**

14. The zeros of $f(x)=x(x-4)(x+2)$ are -2, 0, and 4. Because the graph of $y=f(2x)$ is the graph of $y=f(x)$ compressed horizontally by a factor of $\frac{1}{2}$, the zeros of $y=f(2x)$ are $\frac{1}{2}(-2)=-1$, $\frac{1}{2}(0)=0$, and $\frac{1}{2}(4)=2$. **(3 marks)**

Section 2: Graphing

Key Concepts

- combinations of transformations
- reciprocal function
- absolute value function

The basic goals of this section are to ensure that students are able to

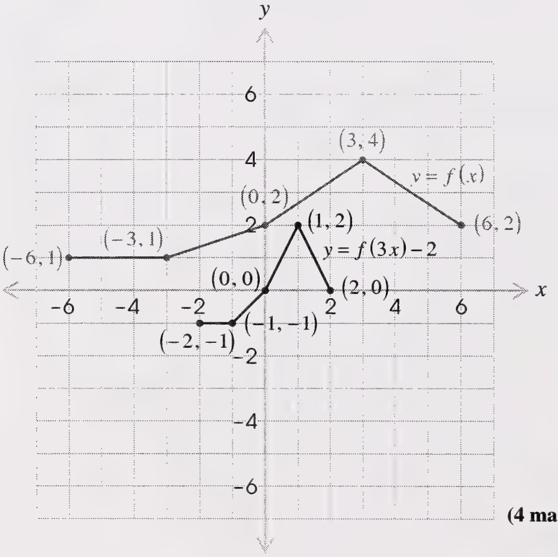
- describe and perform single transformations and combinations of transformations on functions and relations
- describe and draw the graph of $\frac{1}{f(x)}$ using the graph and/or the equation of $f(x)$
- describe and draw the graph of $|f(x)|$ using the graph and/or the equation of $f(x)$

Section 2: Assignment Answer Key (40 marks)

1. a. The graph of $y=-2f(x-1)+3$ is the graph of $y=f(x)$ expanded vertically by a factor of 2, then reflected in the x-axis, and then translated 1 unit to the right and 3 units up. **(4 marks)**

   b. The graph of $y=2f(-3x+12)$ or $y=2f(-3(x-4))$ is the graph of $y=f(x)$ expanded vertically by a factor of 2, then compressed horizontally by a factor of $\frac{1}{3}$, then reflected in the y-axis, and then translated 4 units to the right. **(4 marks)**

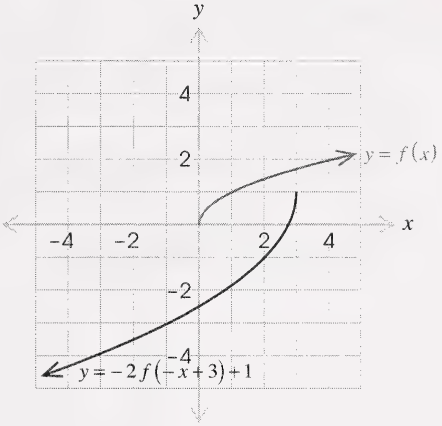
2. The graph of $y=f(3x)-2$ is the graph of $y=f(x)$ compressed horizontally by a factor of $\frac{1}{3}$ and then translated 2 units down.



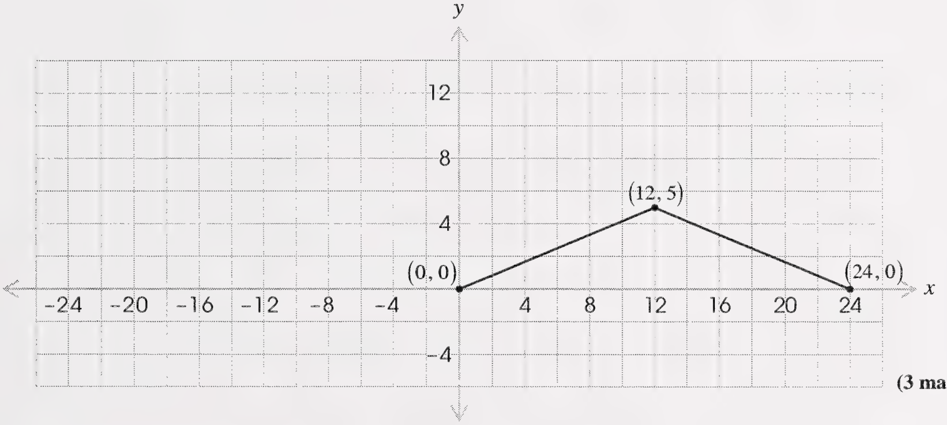
(4 marks)

3. Because $y=-2f(-x+3)+1$ is the same as $y=-2f(-(x-3))+1$ and $f(x)=\sqrt{x}$, graph $y=-2\sqrt{-(x-3)}+1$.

Now, $y=-2\sqrt{-(x-3)}+1$ is the graph of $y=\sqrt{x}$ stretched vertically by a factor of 2, then reflected in both the *x*-axis and the *y*-axis, and then translated 3 units to the right and 1 unit up. **(4 marks)**



4. **a.**



(3 marks)

b. Use the Pythagorean Theorem to find the length of each side of the isosceles triangle.

$$\begin{aligned}\text{side} &= \sqrt{12^2+5^2}\\ &= \sqrt{144+25}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

The height is 5 units; the base is 24 units; and the length of each side is 13 units. **(3 marks)**

c. The slope of the left side is $\frac{5}{12}$, and the slope of the right side is $-\frac{5}{12}$. **(2 marks)**

d. The graph of $y=-\frac{5}{12}|x-12|+5$ is the graph of $y=|x|$ compressed vertically by a factor of $\frac{5}{12}$, then reflected in the *x*-axis, and then translated 12 units to the right and 5 units up.

The graph of $y=-\frac{5}{12}|x-12|+5$, where $y\geq 0$, is the graph of $y=-\frac{5}{12}|x-12|+5$ that lies on or above the *x*-axis. **(4 marks)**

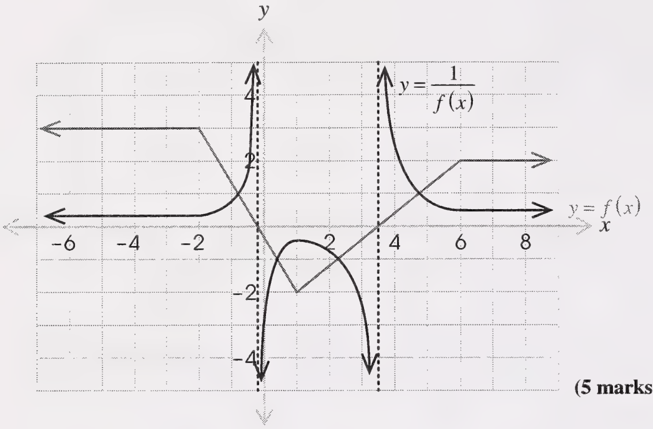
5. When $y = f(x)$ is compressed vertically by a factor of $\frac{1}{2}$, it becomes $y = \frac{1}{2}f(x)$.

When $y = \frac{1}{2}f(x)$ is reflected in the x-axis, it becomes $y = -\frac{1}{2}f(x)$.

When $y = -\frac{1}{2}f(x)$ is translated 4 units to the right and 5 units down, it becomes $y = -\frac{1}{2}f(x-4)-5$.

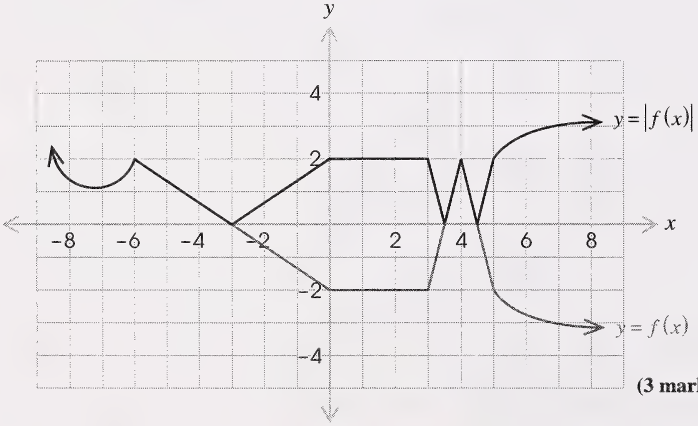
Because $f(x) = |x|$, the transformed function is $g(x) = -\frac{1}{2}|x-4|-5$. **(4 marks)**

6.



(5 marks)

7.



(3 marks)

Final Module Assignment Answer Key (20 marks)

1. Because $y = f(-2x+6)+7$ is the same as $y = f(-2(x-3))+7$, the graph of this function is the graph of $y = f(x)$ compressed horizontally by a factor of $\frac{1}{2}$, then reflected in the y-axis, and then translated 3 units to the right and 7 units up. **(3 marks)**

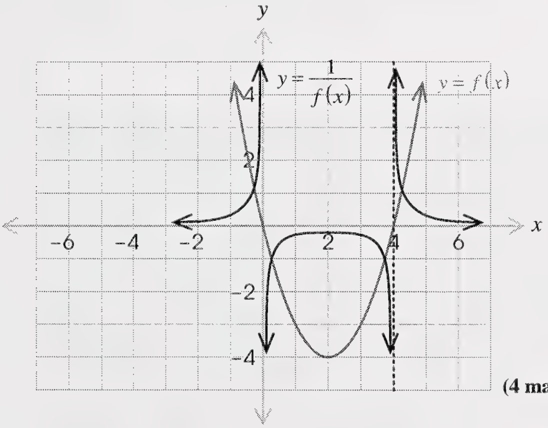
2. When $y = f(x)$ is expanded horizontally by a factor of 2, it becomes $y = f\left(\frac{1}{2}x\right)$.

When $y = f\left(\frac{1}{2}x\right)$ is reflected in the y-axis, it becomes $y = f\left(-\frac{1}{2}x\right)$.

When $y = f\left(-\frac{1}{2}x\right)$ is translated 3 units to the left and 1 unit down, it becomes $y = f\left(-\frac{1}{2}(x+3)\right) - 1$.

Because $f(x) = x^3$, the equation of the translated graph is $y = \left[-\frac{1}{2}(x+3)\right]^3 - 1$ or $y = -\frac{1}{8}(x+3)^3 - 1$. **(3 marks)**

3. **a.**



(4 marks)

b. Use the zeros of $f(x) = x^2 - 4x$ to determine the vertical asymptotes of the graph of $y = \frac{1}{f(x)}$.

Let $f(x) = 0$.

$$\therefore f(x) = x^2 - 4x$$
$$0 = x(x-4)$$

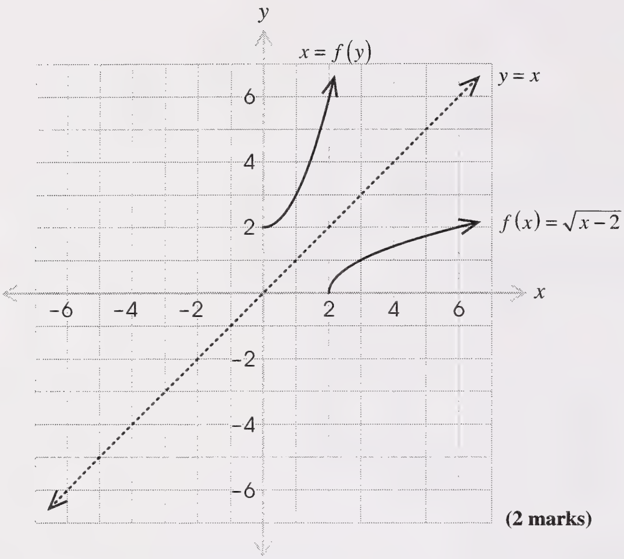
$$\therefore x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 4$$

Therefore, the vertical asymptotes of the graph of $y = \frac{1}{f(x)}$ are $x = 0$ and $x = 4$.

Because $f(x) = x^2 - 4x$ is positive when $x < 0$ or $x > 4$, $\frac{1}{f(x)}$ is positive for these values of x as well. Also, because $f(x)$ is negative when $0 < x < 4$, $\frac{1}{f(x)}$ is negative for these values of x.

Finally, when $|x|$ becomes large, $|x^2 - 4x|$ becomes large and $\frac{1}{f(x)}$ approaches 0. Therefore, the x-axis is a horizontal asymptote. **(4 marks)**

4. **a.**



(2 marks)

b. The reflection in the line $y = x$ is the inverse of $f(x) = \sqrt{x-2}$ or $y = \sqrt{x-2}$.

Interchange the variables to obtain the inverse. Because $\sqrt{}$ represents the principal (or non-negative) square root, $x \geq 0$.

$$x = \sqrt{y-2},\ x \geq 0$$

Now, square both sides, and solve for *y* in terms of *x*.

$$x^2 = \left(\sqrt{y-2}\right)^2,\ x \geq 0$$

$$x^2 = y - 2,\ x \geq 0$$

$$x^2 + 2 = y,\ x \geq 0$$

$$y = x^2 + 2,\ x \geq 0$$

$\therefore\ g(x) = x^2 + 2,\ x \geq 0$ **(3 marks)**

c. Yes, $y = g(x)$ is a function because it passes the vertical line test. For each *x*-value in the domain, there is exactly one *y*-value in the range. **(1 mark)**

Module 2: Exponents and Logarithms

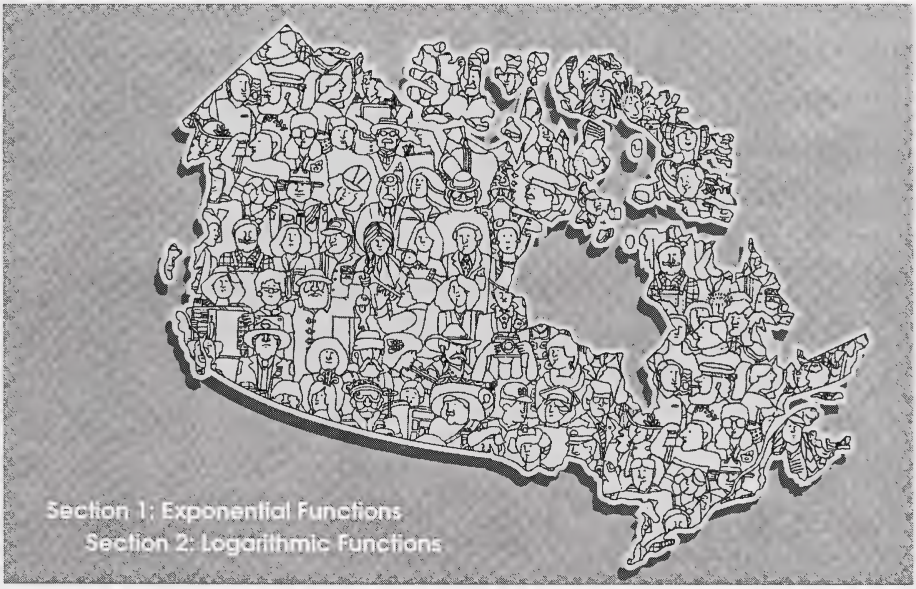
Overview

This module introduces students to exponential and logarithmic functions.

In Section 1, students explore the exponential function. First, students graph and analyse exponential functions using technology. Then students apply exponential functions to solve practical problems, such as compound interest, population growth, and radioactive decay. Students solve the exponential equations arising from these problems using a common base or using graphs and technology.

In Section 2, students explore the logarithmic function. First, students graph and analyse the logarithmic function as the inverse of the exponential function. Then students explore the laws of logarithms and apply them to solve practical problems involving growth and decay. In addition, students gain experience in proving exponential and logarithmic identities and in solving both logarithmic equations and exponential equations using logarithms.

Module 2: Exponents and Logarithms


Section 1: Exponential Functions
Section 2: Logarithmic Functions

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 2A	
Section 1 Assignment	**35 marks**
Assignment Booklet 2B	
Section 2 Assignment	**50 marks**
Final Module Assignment	**15 marks**
TOTAL	**100 marks**

Section 1: Exponential Functions

Key Concepts

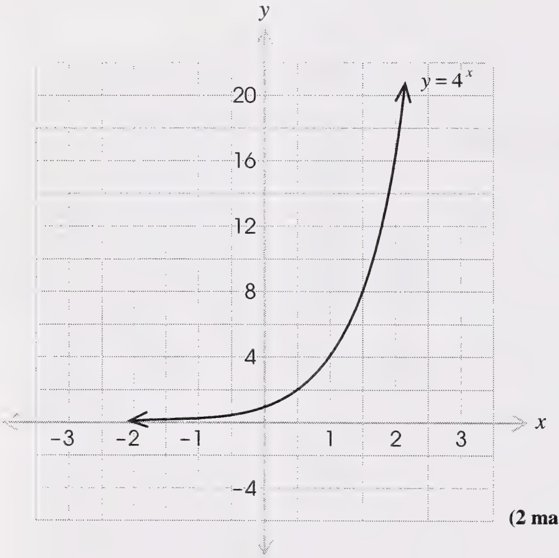
- exponential function
- exponential growth and decay
- exponential equation
- doubling period and half-life

The basic goals of this section are to ensure that students are able to

- graph and analyse an exponential function using technology
- model, graph, and apply exponential functions to solve problems
- solve exponential equations with bases that are powers of one another

Section 1: Assignment Answer Key (35 marks)

1. a.



(2 marks)

b. The domain is the set of real numbers.
The range is $y > 0$.
There is no x-intercept.
The y-intercept is 1.
The horizontal asymptote is $y = 0$.
There is no vertical asymptote.
(4 marks)

c. The graph of $y = \left(\frac{1}{4}\right)^x$ is the reflection of the graph of $y = 4^x$ in the y-axis. Therefore, $b = \frac{1}{4}$ or 0.25.

The equation $y = \left(\frac{1}{4}\right)^x$ can also be written as $y = 4^{-x}$. **(2 marks)**

2. The graph of $y = c^x$ sweeps downward from the second quadrant into the first quadrant; therefore, $0 < c < 1$.

The graphs of $y = a^x$ and $y = b^x$ are both increasing functions; therefore, $a > 1$ and $b > 1$. But because the graph of $y = b^x$ is steeper than the graph of $y = a^x$, $b > a$.

Thus, arranged from smallest to largest, the bases are *c*, *a*, *b*. **(4 marks)**

3. **a.** Use the formula for compound interest, $A(t) = P(l + i)^n$, where $A(t)$ is the amount of the investment after *t* years, *P* is the value of the original investment, *i* is the interest per period, and *n* is the number of interest periods.

Because the interest is compounded quarterly, the interest rate per quarter is $i = \frac{12\%}{4} = 3\% = 0.03$.

Because there are 4 compounding periods per year, the number of periods in *t* years is $4t$. Therefore, $n = 4t$.

$\therefore\ A(t) = 1500(1.03)^{4t}$ **(2 marks)**

b. After 35 years, $t = 35$.

$$\begin{aligned}\therefore\ A(t) &= 1500(1.03)^{4t}\\ A(35) &= 1500(1.03)^{4(35)}\\ &= 1500(1.03)^{140}\\ &= \$94\ 037.86\end{aligned}$$

After 35 years, the investment is worth $94 037.86. **(2 marks)**

4. **a.** Initially, $t = 0$.

$$\begin{aligned}\therefore\ N(t) &= 2500 \times 2^{\frac{t}{d}}\\ N(0) &= 2500 \times 2^{\frac{0}{d}}\\ &= 2500 \times 2^0\\ &= 2500 \times 1\\ &= 2500\end{aligned}$$

There were 2500 bacteria in the initial count. **(2 marks)**

b. $N(t) = 10\ 000$ and $t = 45$

$$\begin{aligned} \therefore\ N(t) &= 2500 \times 2^{\frac{t}{d}} \\ 10\ 000 &= 2500 \times 2^{\frac{45}{d}} \\ 4 &= 2^{\frac{45}{d}} \\ 2^2 &= 2^{\frac{45}{d}} \\ 2 &= \frac{45}{d} \\ 2d &= 45 \\ d &= 22.5 \end{aligned}$$

The doubling period is 22.5 minutes. **(3 marks)**

5. **a.** The equation that models radioactive decay is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$, where $A(t)$ is the amount after t hours, A_0 is the initial sample size, and H is the half-life.

Because $A_0 = 50$ and $H = 14.9$, the equation is $A(t) = 50\left(\frac{1}{2}\right)^{\frac{t}{14.9}}$. **(2 marks)**

b.

$$\begin{aligned} A(t) &= 50\left(\frac{1}{2}\right)^{\frac{t}{14.9}} \\ A(24) &= 50\left(\frac{1}{2}\right)^{\frac{24}{14.9}} \\ &\doteq 16.4 \end{aligned}$$

After 24 h, approximately 16.4 mg of sodium-24 remains. **(2 marks)**

c.

$$\begin{aligned} A(t) &= 12.5\% \text{ of } A_0 \\ &= \frac{12.5}{100} \times A_0 \\ &= \frac{1}{8}A_0 \end{aligned}$$

$$\begin{aligned} \therefore\ A(t) &= A_0\left(\frac{1}{2}\right)^{\frac{t}{14.9}} \\ \frac{1}{8}A_0 &= A_0\left(\frac{1}{2}\right)^{\frac{t}{14.9}} \\ \frac{1}{8} &= \left(\frac{1}{2}\right)^{\frac{t}{14.9}} \\ \left(\frac{1}{2}\right)^3 &= \left(\frac{1}{2}\right)^{\frac{t}{14.9}} \\ 3 &= \frac{t}{14.9} \\ t &= 3(14.9) \\ &= 44.7 \end{aligned}$$

Only 12.5% of the original amount remains after 44.7 h. **(4 marks)**

6. **a.**

$$4^{8x-1} = 8$$
$$\left(2^2\right)^{8x-1} = 2^3$$
$$2^{16x-2} = 2^3$$
$$16x - 2 = 3$$
$$16x = 5$$
$$x = \frac{5}{16}$$

(3 marks)

b.

$$3^{2x-5} = \frac{1}{27^x}$$
$$3^{2x-5} = \frac{1}{\left(3^3\right)^x}$$
$$3^{2x-5} = \frac{1}{3^{3x}}$$
$$3^{2x-5} = 3^{-3x}$$
$$2x - 5 = -3x$$
$$5x = 5$$
$$x = 1$$

(3 marks)

Section 2: Logarithmic Functions

Key Concepts

- logarithmic function
- laws of logarithms
- exponential and logarithmic identities
- logarithmic and exponential form
- common logarithm
- logarithmic equation
- change of base

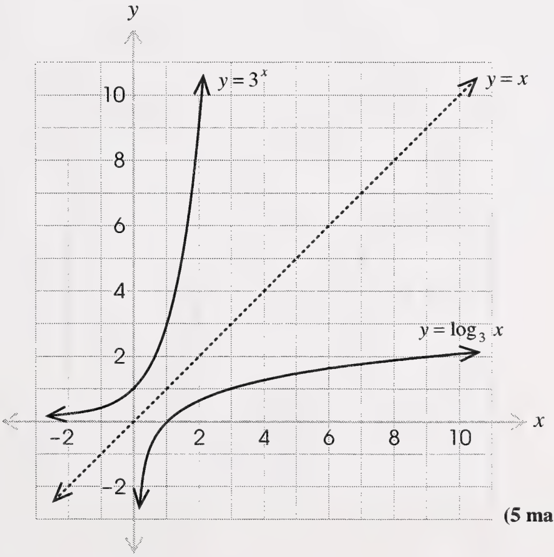
The basic goals of this section are to ensure that students are able to

- graph and analyse the logarithmic function with and without technology
- change functions from exponential form to logarithmic form and vice versa
- explain the relationship between the laws of logarithms and the laws of exponents
- use logarithms to model practical problems
- solve and verify exponential and logarithmic equations and identities

Section 2: Assignment Answer Key (50 marks)

1. **a.** The reflection of the graph of $y = 3^x$ in the line $y = x$ is the inverse function. Therefore, the inverse function is as follows:

$$x = 3^y$$
$$y = \log_3 x$$



(5 marks)

b.

Graph	$y = 3^x$	$y = \log_3 x$
Domain	Reals	$x > 0$
Range	$y > 0$	Reals
***x*-intercept**	none	1
***y*-intercept**	1	none
Horizontal Asymptote	$y = 0$	none
Vertical Asymptote	none	$x = 0$

(4 marks)

2. The graph of $y = \log_c x$ decreases as *x* increases; therefore, $0 < c < 1$.

The graphs of $y = \log_a x$ and $y = \log_b x$ increase as *x* increases; therefore, $a > 1$ and $b > 1$. But because $y = \log_a x$ is steeper than $y = \log_b x$, $b > a$.

Thus, the bases are *c*, *a*, and *b*, arranged from smallest to largest. **(4 marks)**

3. $4^{-\frac{1}{2}} = 0.5$

$\log_4 0.5 = -\frac{1}{2}$ **(1 mark)**

4. $\log_5 \sqrt{5} = 0.5$

$5^{0.5} = \sqrt{5}$ **(1 mark)**

5. a.

$$x = \log_{36} 6$$
$$6 = 36^x$$
$$6 = (6^2)^x$$
$$6^1 = 6^{2x}$$
$$1 = 2x$$
$$x = \frac{1}{2}$$
(2 marks)

b.

$$\log_x 25 = -0.5$$
$$x^{-0.5} = 25$$
$$(x^{-0.5})^{-2} = 25^{-2}$$
$$x^1 = \frac{1}{25^2}$$
$$x = \frac{1}{625}$$
(2 marks)

c.

$$\log_4 x = 2.5$$
$$x = 4^{2.5}$$
$$= 4^{\frac{5}{2}}$$
$$= (\sqrt{4})^5$$
$$= 2^5$$
$$= 32$$
(2 marks)

6. **a.** $$\frac{I_{\text{unknown}}}{I_{\text{Japan}}} = 0.01$$

$$\frac{I_{\text{unknown}}}{10^{8.9}} = 0.01$$

$$\begin{aligned} I_{\text{unknown}} &= 10^{8.9} \times 0.01 \\ &= 10^{8.9} \times 10^{-2} \\ &= 10^{6.9} \end{aligned}$$

The earthquake would read 6.9 on the Richter scale. **(2 marks)**

b. $$\begin{aligned} \frac{I_{\text{Japan}}}{I_{\text{Peru}}} &= \frac{10^{8.9}}{10^{7.7}} \\ &= 10^{1.2} \\ &\doteq 15.8 \end{aligned}$$

The earthquake in Japan was approximately 15.8 times more intense than the earthquake in Peru. **(2 marks)**

7. $$\begin{aligned} \log_2 6 - \log_2 3 + 2\log_2 \sqrt{8} &= \log_2\left[\frac{6\left(\sqrt{8}\right)^2}{3}\right] \\ &= \log_2\left[\frac{6(8)}{3}\right] \\ &= \log_2 16 \quad \longleftarrow \text{single logarithm} \\ &= \log_2 2^4 \\ &= 4 \quad \textbf{(3 marks)} \end{aligned}$$

8. $\log y = \log(0.5x - 3) + \log 2$

Because logarithms of numbers equal to or less than zero are undefined,

$$\begin{aligned} y > 0 \text{ and } 0.5x - 3 &> 0 \\ 0.5x &> 3 \\ x &> 6 \end{aligned}$$

The domain is $x > 6$.

Combine the logarithms on the right side as a single logarithm; then solve for y.

$$\begin{aligned} \log y &= \log(0.5x - 3) + \log 2 \\ \log y &= \log[(0.5x - 3)(2)] \\ \log y &= \log(x - 6) \\ y &= x - 6, \text{ where } x > 6 \end{aligned}$$

(3 marks)

9. $\log_c\left(\frac{\sqrt[3]{a}c^2}{b\sqrt{d}}\right) = \log_c \sqrt[3]{a} + \log_c c^2 - \left(\log_c b + \log_c \sqrt{d}\right)$

$$= \log_c a^{\frac{1}{3}} + \log_c c^2 - \log_c b - \log_c d^{\frac{1}{2}}$$

$$= \frac{\log_c a}{3} + 2 - \log_c b - \frac{\log_c d}{2} \text{ or } \frac{1}{3}\log_c a + 2 - \log_c b - \frac{1}{2}\log_c d$$ **(3 marks)**

10.

$$\begin{aligned} \text{pH} &= -\log\left[\text{H}^+\right] \\ 2.5 &= -\log\left[\text{H}^+\right] \\ \log\left[\text{H}^+\right] &= -2.5 \\ \left[\text{H}^+\right] &= 10^{-2.5} \\ &\doteq 3.2 \times 10^{-3} \end{aligned}$$

The hydrogen ion concentration is approximately 3.2×10^{-3} M/L. **(2 marks)**

11. The equation that models radioactive decay is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$, where $A(t)$ is the amount remaining after t years, A_0 is the initial amount, and H is the half-life.

$A(t) = 12.5$, $A_0 = 16$, and $t = 5$

$$\begin{aligned} \therefore A(t) &= A_0\left(\frac{1}{2}\right)^{\frac{t}{H}} \\ 12.5 &= 16\left(\frac{1}{2}\right)^{\frac{5}{H}} \\ 12.5 &= 16(0.5)^{\frac{5}{H}} \\ \log 12.5 &= \log\left[16(0.5)^{\frac{5}{H}}\right] \\ \log 12.5 &= \log 16 + \frac{5}{H}\log 0.5 \\ \log 12.5 - \log 16 &= \frac{5}{H}\log 0.5 \\ (\log 12.5 - \log 16)H &= 5\log 0.5 \\ H &= \frac{5\log 0.5}{\log 12.5 - \log 16} \\ &\doteq 14.0 \end{aligned}$$

The half-life is approximately 14.0 years. **(4 marks)**

12. $\log_2(x-1) = 5 - \log_2(x+3)$ ← Notice that $x > 1$.

$$\begin{aligned}
\log_2(x-1) + \log_2(x+3) &= 5 \\
\log_2\left[(x-1)(x+3)\right] &= 5 \\
(x-1)(x+3) &= 2^5 \\
x^2 + 2x - 3 &= 32 \\
x^2 + 2x - 35 &= 0 \\
(x+7)(x-5) &= 0
\end{aligned}$$

$\therefore\ x + 7 = 0$ or $x - 5 = 0$

$x = -7$ $x = 5$

Since $x > 1$, the solution is $x = 5$. **(4 marks)**

13.

LS	RS
$\frac{1}{\log a} + \frac{1}{\log a}$	$\frac{1}{\log_{100} a}$
$= \frac{2}{\log a}$	$= \frac{1}{\frac{\log a}{\log 100}}$
	$= \frac{1}{\frac{\log a}{2}}$
	$= \frac{2}{\log a}$

LS = RS

$$\therefore\ \frac{1}{\log a} + \frac{1}{\log a} = \frac{1}{\log_{100} a}$$ **(4 marks)**

14. Let $x = \log_5 11$.

$$\begin{aligned}
\therefore\ 5^x &= 11 \\
\log 5^x &= \log 11 \\
x \log 5 &= \log 11 \\
x &= \frac{\log 11}{\log 5} \\
&\doteq 1.49
\end{aligned}$$ **(2 marks)**

Final Module Assignment Answer Key (15 marks)

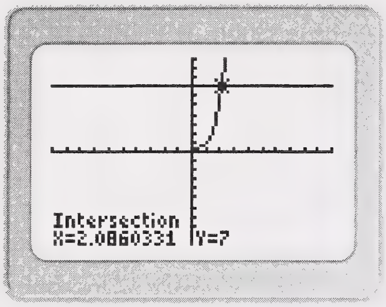
1. **a.** The system of equations you could enter to solve $6^{x-1} = 7$ is as follows:

$y = 6^{x-1}$ ①

$y = 7$ ②

(1 mark)

b.



Intersection
X=2.0860331 Y=7

Therefore, $x \doteq 2.086$. **(4 marks)**

c.

$$\begin{aligned} 6^{x-1} &= 7 \\ \log 6^{x-1} &= \log 7 \\ (x-1)\log 6 &= \log 7 \\ x\log 6 - \log 6 &= \log 7 \\ x\log 6 &= \log 7 + \log 6 \\ x &= \frac{\log 7 + \log 6}{\log 6} \\ &\doteq 2.086 \end{aligned}$$

(4 marks)

2.

$$\begin{aligned} N(t) &= N_0 (2)^{\frac{t}{d}} \\ \log N(t) &= \log\left[N_0 \times (2)^{\frac{t}{d}}\right] \\ \log N(t) &= \log N_0 + \frac{t}{d}\log 2 \\ \log N(t) - \log N_0 &= \frac{t}{d}\log 2 \\ d\left(\log N(t) - \log N_0\right) &= t\log 2 \\ t &= \frac{d\left(\log N(t) - \log N_0\right)}{\log 2} \end{aligned}$$

(4 marks)

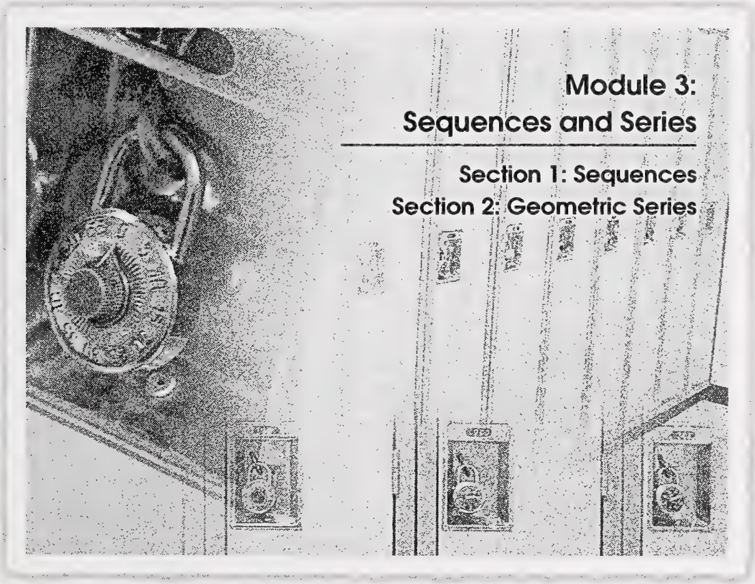
3. $$\begin{aligned}\log_2 16x^2 &= \log_2 16 + \log_2 x^2\\ &= \log_2 2^4 + 2\log_2 x\\ &= 4 + 2\log_2 x\\ &= 4 + 2(7)\\ &= 4 + 14\\ &= 18 \quad \textbf{(2 marks)}\end{aligned}$$

Module 3: Sequences and Series

Overview

In Section 1, students begin their exploration of sequences by reviewing the concepts and terminology from previous studies. They practise generating sequences using general terms and recursive definitions. Then students explore, in detail, geometric sequences—the focus of this section. In particular, they investigate the connection between geometric sequences and geometric growth using the principles and processes of exponents and logarithms.

In Section 2, students explore both finite and infinite geometric series. In this section, students recognize and apply the condition for the convergence of infinite geometric series in both theoretical and practical problems.



Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 3A	
Section 1 Assignment	**30 marks**
Assignment Booklet 3B	
Section 2 Assignment	**30 marks**
Final Module Assignment	**40 marks**
TOTAL	**100 marks**

Section 1: Sequences

Key Concepts

- general term
- explicit formula
- recursive formula
- geometric sequence
- common ratio
- geometric growth

The basic goals of this section are to ensure that students are able to

- recognize and generate the terms of a geometric sequence
- connect geometric sequences to exponential functions over the natural numbers
- derive and apply expressions to represent general terms for geometric growth and solve problems

Section 1: Assignment Answer Key (30 marks)

1. **a.** The sequence is geometric.

$$a = 600 \text{ and } r = \frac{300}{600} = \frac{1}{2}$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ &= 600\left(\frac{1}{2}\right)^{n-1} \quad \textbf{(2 marks)} \end{aligned}$$

b. The sequence is arithmetic.

$$a = -12 \text{ and } d = 4$$

$$\begin{aligned} \therefore\ t_n &= a + (n-1)d \\ &= -12 + (n-1)4 \\ &= -12 + 4n - 4 \\ &= 4n - 16 \quad \textbf{(2 marks)} \end{aligned}$$

2. $a = \frac{4}{3}$ and $r = \frac{\frac{4}{9}}{\frac{4}{3}} = \frac{1}{3}$

$$\begin{aligned}
\therefore\ t_n &= ar^{n-1} \\
&= \frac{4}{3}\left(\frac{1}{3}\right)^{n-1} \quad \textbf{(1 mark)} \\
t_{10} &= \frac{4}{3}\left(\frac{1}{3}\right)^{10-1} \\
&= \frac{4}{3}\left(\frac{1}{3}\right)^{9} \\
&= \frac{4}{3^{10}} \\
&= \frac{4}{59\ 049} \quad \textbf{(2 marks)}
\end{aligned}$$

3.

$$\begin{aligned}
t_n &= 5(2)^{n-1} \\
t_1 &= 5(2)^{1-1} \\
&= 5(2)^0 \\
&= 5(1) \\
&= 5
\end{aligned}
\qquad
\begin{aligned}
t_n &= 5(2)^{n-1} \\
t_2 &= 5(2)^{2-1} \\
&= 5(2)^1 \\
&= 5(2) \\
&= 10
\end{aligned}
\qquad
\begin{aligned}
t_n &= 5(2)^{n-1} \\
t_3 &= 5(2)^{3-1} \\
&= 5(2)^2 \\
&= 5(4) \\
&= 20
\end{aligned}$$

The first three terms are 5, 10, and 20. **(2 marks)**

$$\begin{aligned}
\frac{t_n}{t_{n-3}} &= \frac{\cancel{5}(2)^{n-1}}{\cancel{5}(2)^{n-3-1}} \\
&= \frac{2^{n-1}}{2^{n-4}} \\
&= 2^{n-1-(n-4)} \\
&= 2^{n-1-n+4} \\
&= 2^3 \\
&= 8 \quad \textbf{(2 marks)}
\end{aligned}$$

4. Starting with the second term, each term is 3 times larger than the preceding term.

$\therefore\ t_n = 3t_{n-1},\ n > 1$

The recursive definition of this geometric sequence is $t_n = 3t_{n-1}$, $n > 1$ and $t_1 = 1$. **(2 marks)**

5. At the end of the first year, the car is worth

$$\begin{aligned} 80\% \text{ of } 25\,000 &= 0.80 \times 25\,000 \\ &= \$20\,000 \end{aligned}$$

At the end of each successive year, the car is worth 90% of its previous value. This is an example of a geometric series, where $a = 20\,000$, $r = 0.90$, and $n = 10$.

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ t_{10} &= 20\,000(0.90)^{10-1} \\ &= 20\,000(0.90)^{9} \\ &\doteq 7750 \end{aligned}$$

After 10 years, the car is worth about \$7750. **(3 marks)**

6. $r = \dfrac{t_2}{t_1}$

$= \dfrac{2x+8}{18x-9}$ ①

$r = \dfrac{t_3}{t_2}$

$= \dfrac{x-1}{2x+8}$ ②

Substitute $\dfrac{2x+8}{18x-9}$ for r into ②.

$$\begin{aligned} r &= \frac{x-1}{2x+8} \\ \frac{2x+8}{18x-9} &= \frac{x-1}{2x+8} \\ (2x+8)(2x+8) &= (x-1)(18x-9) \\ 4x^2 + 16x + 16x + 64 &= 18x^2 - 9x - 18x + 9 \\ 4x^2 + 32x + 64 &= 18x^2 - 27x + 9 \\ 0 &= 14x^2 - 59x - 55 \end{aligned}$$

$$\begin{aligned} \therefore\ x &= \frac{-(-59) \pm \sqrt{(-59)^2 - 4(14)(-55)}}{2(14)} \\ &= \frac{59 \pm \sqrt{3481 + 3080}}{28} \\ &= \frac{59 \pm \sqrt{6561}}{28} \\ &= \frac{59 \pm 81}{28} \end{aligned}$$

$$\begin{aligned} \therefore\ x &= \frac{59+81}{28} \\ &= \frac{140}{28} \\ &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \frac{59-81}{28} \\ &= \frac{-22}{28} \\ &= -\frac{11}{14} \end{aligned}$$

One geometric series is as follows:

$$\begin{aligned} 18x-9,\ 2x+8,\ x-1,\ldots &= 18(5)-9,\ 2(5)+8,\ 5-1,\ldots \\ &= 90-9,\ 10+8,\ 4,\ldots \\ &= 81,\ 18,\ 4,\ldots \end{aligned}$$

The other geometric series is as follows:

$$\begin{aligned} 18x-9,\ 2x+8,\ x-1,\ldots &= \overset{9}{\cancel{18}}\left(-\frac{11}{\underset{7}{\cancel{14}}}\right)-9,\ \overset{1}{\cancel{2}}\left(-\frac{11}{\underset{7}{\cancel{14}}}\right)+8,\ -\frac{11}{14}-1,\ldots \\ &= -\frac{99}{7}-9,\ -\frac{11}{7}+8,\ -\frac{11}{14}-\frac{14}{14},\ldots \\ &= -\frac{99}{7}-\frac{63}{7},\ -\frac{11}{7}+\frac{56}{7},\ -\frac{25}{14},\ldots \\ &= -\frac{162}{7},\ \frac{45}{7},\ -\frac{25}{14},\ldots \end{aligned}$$

(5 marks)

7. Determine which term is the middle term.

$$a = 0.1024$$

$$r = \frac{0.256}{0.1024} = 2.5$$

$$t_n = 156.25$$

$$\begin{aligned} \therefore\ t_n &= ar^{n-1} \\ 156.25 &= 0.1024(2.5)^{n-1} \\ \frac{156.25}{0.1024} &= 2.5^{n-1} \\ 2.5^8 &= 2.5^{n-1} \\ 8 &= n-1 \\ n &= 9 \end{aligned}$$

Because the ninth term is the middle term, there are 17 terms in the sequence. **(3 marks)**

8. **a.** $t_1 = 5$

$$\begin{array}{llll} t_n = 2t_{n-1} & t_n = 2t_{n-1} & t_n = 2t_{n-1} & t_n = 2t_{n-1} \\ t_2 = 2t_{2-1} & t_3 = 2t_{3-1} & t_4 = 2t_{4-1} & t_5 = 2t_{5-1} \\ \quad = 2t_1 & \quad = 2t_2 & \quad = 2t_3 & \quad = 2t_4 \\ \quad = 2(5) & \quad = 2(10) & \quad = 2(20) & \quad = 2(40) \\ \quad = 10 & \quad = 20 & \quad = 40 & \quad = 80 \end{array}$$

The first five terms are 5, 10, 20, 40, and 80. **(1 mark)**

b. The points would lie on the graph of an exponential function. Geometric sequences, where $r > 0$ and $r \neq 1$, have points that lie on an exponential curve. **(2 marks)**

9. $t_n = ar^{n-1}$ $\qquad$ $t_n = ar^{n-1}$

$t_3 = ar^{3-1}$ $\qquad$ $t_9 = ar^{9-1}$

$-3 = ar^2$ ① $\qquad$ $-192 = ar^8$ ②

Divide ② by ①.

$$\frac{-192}{-3} = \frac{ar^8}{ar^2}$$
$$64 = r^6$$
$$r^6 = 64$$
$$r = \sqrt[6]{64}$$
$$= 2$$

Substitute $r = 2$ into ①.

$$-3 = ar^2$$
$$-3 = a(2)^2$$
$$-3 = 4a$$
$$4a = -3$$
$$a = -\frac{3}{4}$$

The first term, a, is $-\frac{3}{4}$. **(3 marks)**

Section 2: Geometric Series

Key Concepts

- geometric series
- finite series
- infinite series
- convergence

The basic goals of this section are to ensure that students are able to

- recognize and evaluate geometric series
- derive and apply expressions to represent sums for geometric growth and solve problems
- estimate values of expressions for infinite geometric processes

Section 2: Assignment Answer Key (30 marks)

1. $a = 8$

$$r = \frac{48}{8} = 6$$

$$\begin{aligned} \therefore S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ S_{10} &= \frac{8\left(1-6^{10}\right)}{1-6} \\ &= \frac{8\left(1-6^{10}\right)}{-5} \\ &= 96\ 745\ 880 \quad \textbf{(3 marks)} \end{aligned}$$

2. **a.** Find the number of terms, n, in the series.

$$a = -\frac{1}{32}$$

$$r = \frac{\frac{1}{16}}{-\frac{1}{32}} = -2$$

$$t_n = 256$$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ 256 &= -\frac{1}{32}(-2)^{n-1} \\ (-2)^8 &= (-2)^{-5}(-2)^{n-1} \\ (-2)^8 &= (-2)^{n-6} \\ 8 &= n-6 \\ n &= 14 \end{aligned}$$

Now find the sum of the series.

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ &= \frac{-\frac{1}{32}\left[1-(-2)^{14}\right]}{1-(-2)} \\ &= \frac{-\frac{1}{32}\left[1-(-2)^{14}\right]}{3} \\ &\doteq 170.6563 \quad \textbf{(3 marks)} \end{aligned}$$

b. $$\begin{aligned} \sum_{n=1}^{50} 8(0.5)^{n-2} &= 8(0.5)^{1-2} + 8(0.5)^{2-2} + \ldots + 8(0.5)^{50-2} \\ &= 8(0.5)^{-1} + 8(0.5)^{0} + \ldots + 8(0.5)^{48} \end{aligned}$$

$$\begin{aligned}\therefore\ S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ &= \frac{8(0.5)^{-1}\left[1-0.5^{50}\right]}{1-0.5}\\ &= \frac{8(0.5)^{-1}\left[1-0.5^{50}\right]}{0.5}\\ &= \frac{8\left(1-0.5^{50}\right)}{0.5^2}\\ &\doteq 32 \quad \textbf{(3 marks)}\end{aligned}$$

c. $$\begin{aligned}\sum_{n=1}^{\infty} 5(0.2)^n &= 5(0.2)^1 + 5(0.2)^2 + 5(0.2)^3 + \ldots\\ &= 1 + 0.2 + 0.04 + \ldots\end{aligned}$$

Because $-1 < r < 1$,

$$\begin{aligned}S_n &= \frac{a}{1-r}\\ &= \frac{1}{1-0.2}\\ &= \frac{1}{0.8}\\ &= 1.25 \quad \textbf{(3 marks)}\end{aligned}$$

3. Determine the number of terms, n, in the series.

$$a = \frac{3}{32}$$

$$r = \frac{\frac{3}{16}}{\frac{3}{32}} = 2$$

$$t_n = 12\ 288$$

$$\begin{aligned}\therefore\ t_n &= ar^{n-1}\\ 12\ 288 &= \frac{3}{32}(2)^{n-1}\\ 12\ 288 &= 3(2)^{-5}(2)^{n-1}\\ 12\ 288 &= 3(2)^{n-6}\\ 4096 &= 2^{n-6}\\ 2^{12} &= 2^{n-6}\\ 12 &= n-6\\ n-6 &= 12\\ n &= 18\end{aligned}$$

Because this is a sum of a geometric series,

$$\begin{aligned}\sum_{k=1}^{n} t_n &= \sum_{k=1}^{n} ar^{k-1}\\ &= \sum_{k=1}^{18} \frac{3}{32}(2)^{k-1} \text{ or } \sum_{k=1}^{18} 3(2)^{k-6} \quad \textbf{(3 marks)}\end{aligned}$$

4. For $t_n = (0.75)^{n-1}$, $a = 1$ and $r = 0.75$.

$$\begin{aligned}\therefore\ S &= \frac{a}{1-r}\\ &= \frac{1}{1-0.75}\\ &= \frac{1}{0.25}\\ &= 4\end{aligned}$$

For $t_n = \left(\frac{3}{8}\right)^{n-1}$, $a = 1$ and $r = \frac{3}{8}$.

$$\begin{aligned}\therefore\ S &= \frac{a}{1-r}\\ &= \frac{1}{1-\frac{3}{8}}\\ &= \frac{1}{\frac{5}{8}}\\ &= \frac{8}{5} \text{ or } 1.6\end{aligned}$$ **(4 marks)**

5. $t_1 = S_1$
$= 0.0016$

$t_2 = S_2 - S_1$
$= 0.0096 - 0.0016$
$= 0.0080$

$t_3 = S_3 - S_2$
$= 0.0496 - 0.0096$
$= 0.0400$

The terms form a geometric sequence.

$\therefore\ a = 0.0016$
$r = \frac{0.0080}{0.0016} = 5$

$\therefore\ t_n = ar^{n-1}$
$= 0.0016(5)^{n-1}$ **(4 marks)**

6. For Graph A, $t_1 = 5$, $t_2 = 3$, $t_3 = 1$, $t_4 = -1$, and $t_s = -3$. This is an arithmetic sequence because the terms are decreasing by 3.

For Graph B, $t_1 = \frac{1}{2}$, $t_2 = 1$, $t_3 = 2$, $t_4 = 4$, and $t_5 = 8$. This is a geometric sequence because each term is 2 times the preceding term.

$$\therefore a = \frac{1}{2}$$

$$r = 2$$

$$\begin{aligned}\therefore S_n &= \frac{a(1-r^n)}{1-r}\\ S_8 &= \frac{\frac{1}{2}(1-2^8)}{1-2}\\ &= \frac{\frac{1}{2}(1-2^8)}{-1}\\ &= -\frac{1}{2}(1-128)\\ &= -\frac{1}{2}(-127)\\ &= \frac{127}{2} \text{ or } 63\frac{1}{2} \quad \textbf{(4 marks)}\end{aligned}$$

7. $a = 5$

$$r = \frac{-\frac{5}{2}}{5} = -\frac{1}{2}$$

$$\begin{aligned}\therefore S &= \frac{a}{1-r}\\ &= \frac{5}{1-\left(-\frac{1}{2}\right)}\\ &= \frac{5}{1+\frac{1}{2}}\\ &= \frac{5}{\frac{3}{2}}\\ &= \frac{10}{3} \text{ or } 3\frac{1}{3} \quad \textbf{(3 marks)}\end{aligned}$$

Final Module Assignment Answer Key (40 marks)

1. **a.** $t_1 = 2$

$$\begin{aligned}t_n &= 5t_{n-1} + 3\\ t_2 &= 5t_{2-1} + 3\\ &= 5t_1 + 3\\ &= 5(2) + 3\\ &= 10 + 3\\ &= 13\end{aligned}$$

$$\begin{aligned}t_n &= 5t_{n-1} + 3\\ t_3 &= 5t_{3-1} + 3\\ &= 5t_2 + 3\\ &= 5(13) + 3\\ &= 65 + 3\\ &= 68\end{aligned}$$

The sequence 2, 13, 68, … is neither arithmetic nor geometric. **(2 marks)**

b.
$$\begin{aligned} t_n &= -3(4)^n \\ t_1 &= -3(4)^1 \\ &= -3(4) \\ &= -12 \end{aligned} \qquad \begin{aligned} t_n &= -3(4)^n \\ t_2 &= -3(4)^2 \\ &= -3(16) \\ &= -48 \end{aligned} \qquad \begin{aligned} t_n &= -3(4)^n \\ t_3 &= -3(4)^3 \\ &= -3(64) \\ &= -192 \end{aligned}$$

The sequence $-12, -48, -192, \ldots$ is geometric. **(2 marks)**

c.
$$\begin{aligned} t_n &= 2+5n \\ t_1 &= 2+5(1) \\ &= 2+5 \\ &= 7 \end{aligned} \qquad \begin{aligned} t_n &= 2+5n \\ t_2 &= 2+5(2) \\ &= 2+10 \\ &= 12 \end{aligned} \qquad \begin{aligned} t_n &= 2+5n \\ t_3 &= 2+5(3) \\ &= 2+15 \\ &= 17 \end{aligned}$$

The sequence 7, 12, 17, … is arithmetic. **(2 marks)**

2.
$$\begin{aligned} t_n &= ar^{n-1} \\ t_3 &= ar^{3-1} \\ -75 &= ar^2 \quad ① \end{aligned} \qquad \begin{aligned} t_n &= ar^{n-1} \\ t_6 &= ar^{6-1} \\ 9375 &= ar^5 \quad ② \end{aligned}$$

Divide ② by ①.

$$\begin{aligned} \frac{9375}{-75} &= \frac{ar^5}{ar^2} \\ -125 &= r^3 \\ r^3 &= -125 \\ r &= \sqrt[3]{-125} \\ &= -5 \end{aligned}$$

Substitute $r = -5$ into ①.

$$\begin{aligned} -75 &= ar^2 \\ -75 &= a(-5)^2 \\ -75 &= 25a \\ 25a &= -75 \\ a &= -3 \end{aligned}$$

The first term is -3, and the common ratio is -5. **(4 marks)**

3. 2, 3, 5, 6, 7, 10, 11, 12, 13, 14, … **(2 marks)**

4. 1st month $= 25\ 000$

2nd month $= 25\ 000(0.95)$

3rd month $= 25\ 000(0.95)^2$

$\vdots \qquad \vdots$

12 th month $= 25\ 000(0.95)^{11}$

The total production is the sum of the geometric series $25\ 000 + 25\ 000(0.95) + 25\ 000(0.95)^2 + \ldots + 25\ 000(0.95)^{11}$.

$\therefore\ a = 25\ 000$ and $r = 0.95$

$$\begin{aligned}\therefore\ S_n &= \frac{a\left(1-r^n\right)}{1-r}\\ S_{12} &= \frac{25\ 000\left(1-0.95^{12}\right)}{1-0.95}\\ &= \frac{25\ 000\left(1-0.95^{12}\right)}{0.05}\\ &= 229\ 819.9562\end{aligned}$$

The oil company will produce about 230 000 barrels. **(4 marks)**

5. $A = 1583.30$

$i = \frac{8\%}{4} = 0.02$

$n = 4 \times 3 = 12$

$$\begin{aligned}\therefore\ A &= P(1+i)^n\\ 1585.30 &= P(1+0.02)^{12}\\ 1585.30 &= P(1.02)^{12}\\ \frac{1585.30}{(1.02)^{12}} &= P\\ P &= 1250.00\end{aligned}$$

Donna deposited \$1250 into the bank account 3 years ago. **(3 marks)**

6. **a.** Income in 1st year $= 3200$

Income in 2nd year $= 3200(1.07)$

$\vdots \qquad \vdots$

Income in 5th year $= 3200(1.07)^4$

Kenneth's gross monthly income in his fifth year at work is $3200(1.07)^4 = \$4194.55$.

Therefore, Kenneth's gross annual income is $4194.55 \times 12 = \$50\ 334.60$. **(4 marks)**

b. To earn at least \$60 000 per year, Kenneth has to earn at least $60\ 000 \div 12 = \$5000$ per month.

$$\therefore\ a = 3200 \qquad r = 1.07 \qquad t_n = 5000$$

$$\begin{aligned}
t_n &= ar^{n-1}\\
5000 &= 3200(1.07)^{n-1}\\
1.5625 &= 1.07^{n-1}\\
\log 1.5625 &= \log 1.07^{n-1}\\
\log 1.5625 &= (n-1)\log 1.07\\
n-1 &= \frac{\log 1.5625}{\log 1.07}\\
n &= \frac{\log 1.5625}{\log 1.07}+1\\
&\doteq 7.596
\end{aligned}$$

To earn at least \$60 000 per year, Kenneth would have to work a minimum of 8 years. **(4 marks)**

7. Let P represent the amount the farmer was paid for the bananas.

The cost of the bananas at each stage can be represented by the following geometric sequence:

$$\underset{\text{farmer}}{P},\ \overset{\text{market}}{P(1.20)},\ \underset{\text{wholesaler}}{P(1.20)^2},\ \overset{\text{grocery store}}{P(1.20)^3}$$

$$\begin{aligned}
\therefore\ P(1.20)^3 &= 80\\
P &= \frac{80}{(1.20)^3}\\
&= 46
\end{aligned}$$

The farmer was paid 46¢ for the bananas. **(4 marks)**

8. **a.** $0.\overline{12} = 0.12 + 0.0012 + 0.000\ 012 + \ldots$ **(2 marks)**

b. $a = 0.12$ and $r = 0.01$

$$\begin{aligned}\therefore\ S &= \frac{a}{1-r}\\ &= \frac{0.12}{1-0.01}\\ &= \frac{0.12}{0.99}\\ &= \frac{12}{99}\\ &= \frac{4}{33}\end{aligned}$$

$\therefore\ 0.\overline{12} = \frac{4}{33}$ **(2 marks)**

9. a. Because the general term is defined by $t_n = k(0.3)^{n-1}$,

$$\begin{aligned}t_1 &= k(0.3)^{1-1}\\ &= k(0.3)^0\\ &= k\end{aligned}$$

Therefore, k represents the first term. Because the infinite sum is defined by $S = \frac{a}{1-r}$, for this series, $a = k$ and $r = 0.3$

$$\begin{aligned}\therefore\ S &= \frac{k}{1-0.3}\\ &= \frac{k}{1-0.3}\\ &= \frac{k}{0.7}\\ &= \frac{10}{7}k\end{aligned}$$ **(3 marks)**

b.

$$\begin{aligned}S &= \frac{10}{7}k\\ 90 &= \frac{10}{7}k\\ 10k &= 630\\ k &= 63\end{aligned}$$ **(2 marks)**

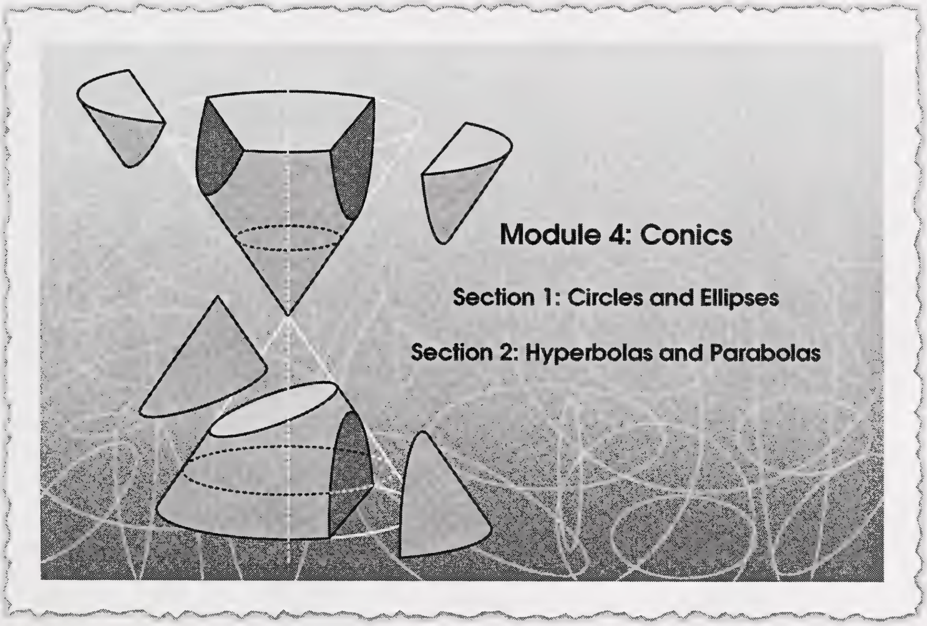
Module 4: Conics

Overview

This module introduces students to the conic sections—the circle, the ellipse, the hyperbola, and the parabola.

In Section 1, students investigate curves that arise from the intersection between a plane and a double-napped, right circular cone and between a plane and a cylinder. Through their investigation, students distinguish between primary conics and their degenerate forms. After this introduction to conics, students focus on the circle and the ellipse. Students begin by analysing circles from their equations, converting between standard form and general form and practising to complete the square. Once they have mastered the circle, students explore the ellipse through transformations of the unit circle. Again, students analyse graphs and derive equations in both standard form and general form.

In Section 2, students investigate the hyperbola and the parabola from a transformational perspective. The hyperbola was introduced by transforming the unit hyperbola, $x^2 - y^2 = 1$; and the parabola was introduced by transforming $y = x^2$. Students work with general form and standard form of equations, convert between forms, identify conics from these forms, and use the equations to sketch and analyse the graphs.


Module 4: Conics
Section 1: Circles and Ellipses
Section 2: Hyperbolas and Parabolas

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 4A	
Section 1 Assignment	**45 marks**
Assignment Booklet 4B	
Section 2 Assignment	**40 marks**
Final Module Assignment	**15 marks**
TOTAL	**100 marks**

Section 1: Circles and Ellipses

Key Concepts

- double-napped cone
- conic section
- degenerate form
- circle
- ellipse
- axis of symmetry
- vertex
- standard form
- general form

The basic goals of this section are to ensure that students are able to

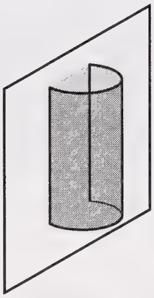
- classify conic sections, using their shapes and equations
- classify and analyse circles and ellipses from their equations using a transformational approach
- convert the equations of circles and ellipses from general form to standard form and vice versa

Section 1: Assignment Answer Key (45 marks)

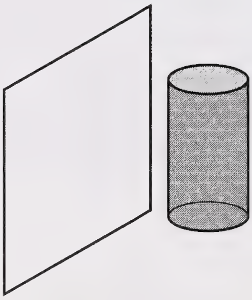
1. **a.** $\theta = 90°$ **(1 mark)**
   b. $\theta = 52.5°$ **(1 mark)**
   c. $52.5° < \theta < 90°$ **(1 mark)**
   d. $0° \leq \theta < 52.5°$ **(1 mark)**

2. To obtain a degenerate hyperbola, the cutting plane must pass through the vertex of the double-napped cone at an angle, θ, with the central axis within $0° \leq \theta < 52.5°$. **(3 marks)**

3. A pair of parallel lines and the case where no curve exists cannot be obtained from the intersection of a plane and a double-napped cone. These degenerate forms are obtained from the intersection of a plane and a cylinder. **(4 marks)**



2 Parallel Lines



No Curve

4. The graph of this circle is the graph of the unit circle stretched horizontally and vertically by a factor of 5, and then translated 3 units to the right and 4 units down. **(4 marks)**

   Note to teacher: Students may describe the circle by noting that the radius of the unit circle has been increased by a factor of 5, rather than by referencing horizontal and vertical stretches. This reasoning is acceptable.

5. **a.**

$$x^2+y^2-6x+8y+9=0$$

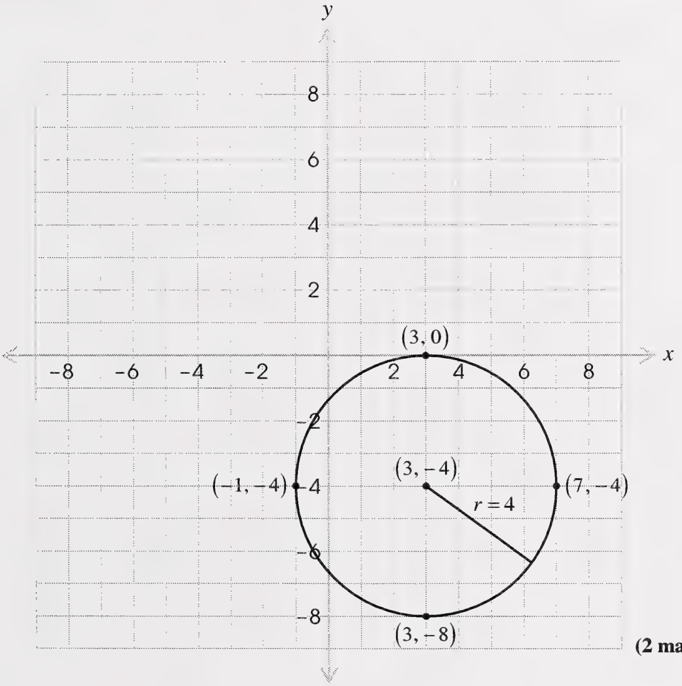
$$x^2-6x+y^2+8y=-9$$

$$\left[\left(x^2-6x+9\right)-9\right]+\left[\left(y^2+8y+16\right)-16\right]=-9$$

$$(x-3)^2-9+(y+4)^2-16=-9$$

$$(x-3)^2+(y+4)^2=16$$ **(3 marks)**

b. Compare $(x-3)^2+(y+4)^2=16$ with $(x-h)^2+(y-k)^2=r^2$. Since $h=3$, $k=-4$, and $r=\sqrt{16}=4$, the centre, (h, k), of the circle is $(3, -4)$ and its radius is 4 units.



(2 marks)

The domain is $-1 \le x \le 7$, and the range is $-8 \le y \le 0$. **(2 marks)**

6. Use $(x-h)^2+(y-k)^2=r^2$. Because the centre, (h,k), of the circle is $(4,-1)$, $h=4$ and $k=-1$.

$$\therefore\ (x-4)^2+[y-(-1)]^2=r^2$$
$$(x-4)^2+(y+1)^2=r^2$$

The point $(0,-2)$ lies on the circle. Substitute 0 for x and -2 for y.

$$(0-4)^2+(-2+1)^2=r^2$$
$$(-4)^2+(-1)^2=r^2$$
$$16+1=r^2$$
$$r^2=17$$

Therefore, the equation of the circle is $(x-4)^2+(y+1)^2=17$. **(4 marks)**

7. **a.** Because $x^2+y^2=1$ has been stretched horizontally by a factor of 3 and vertically by a factor of 4, substitute $\frac{x}{3}$ for x and $\frac{y}{4}$ for y.

$$\therefore\ \left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2=1$$
$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

The equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{16}=1$. **(3 marks)**

b. To translate the ellipse 4 units to the right and 2 units down, substitute $(x-4)$ for x and $(y+2)$ for y.

$$\therefore\ \frac{(x-4)^2}{9}+\frac{(y+2)^2}{16}=1 \quad \longleftarrow \text{ standard form}$$

Express the equation in general form.

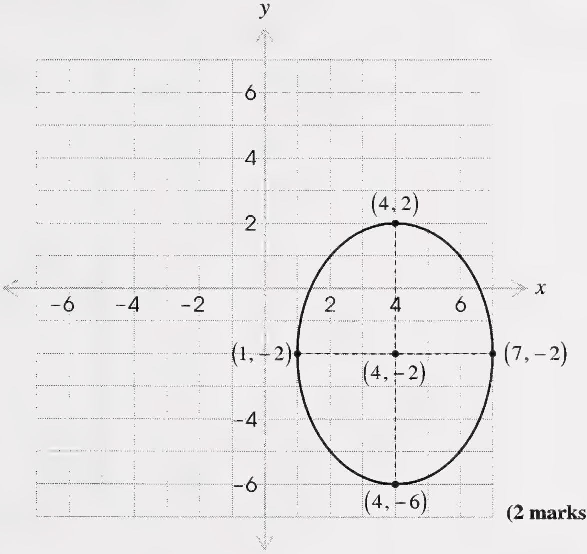
$$16(x-4)^2+9(y+2)^2=9(16)$$
$$16\left(x^2-8x+16\right)+9\left(y^2+4y+4\right)=144$$
$$16x^2-128x+256+9y^2+36y+36=144$$
$$16x^2+9y^2-128x+36y+148=0 \quad \longleftarrow \text{ general form}$$

(4 marks)

c. Compare $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{16}=1$ with $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

$\therefore\ h=4,\ k=-2,\ a=\sqrt{16}=4$, and $b=\sqrt{9}=3$

The centre is at $(4,-2)$ and the major axis is parallel to the y-axis.



(2 marks)

The domain is $1\le x\le 7$, and the range is $-6\le y\le 2$. **(2 marks)**

8. **a.**

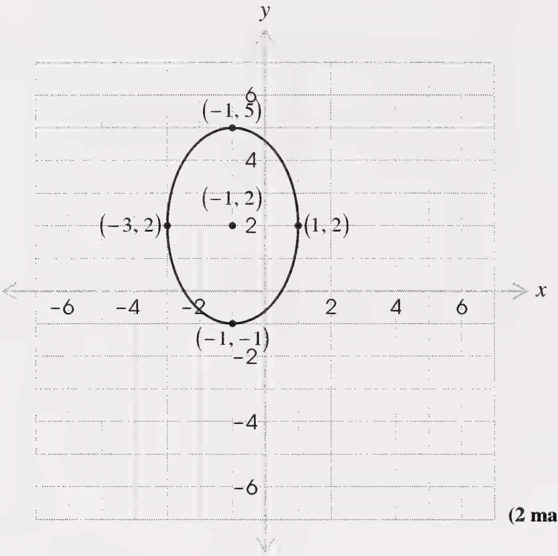
$$\begin{aligned}
9x^2+4y^2+18x-16y-11&=0\\
9x^2+18x+4y^2-16y&=11\\
9\left(x^2+2x\right)+4\left(y^2-4y\right)&=11\\
9\left[\left(x^2+2x+1\right)-1\right]+4\left[\left(y^2-4y+4\right)-4\right]&=11\\
9\left[(x+1)^2-1\right]+4\left[(y-2)^2-4\right]&=11\\
9(x+1)^2-9+4(y-2)^2-16&=11\\
9(x+1)^2+4(y-2)^2&=36\\
\frac{9(x+1)^2}{36}+\frac{4(y-2)^2}{36}&=\frac{36}{36}\\
\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}&=1
\end{aligned}$$

(4 marks)

b. Compare $\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}=1$ with $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

$\therefore\ h=-1,\ k=2,\ a=3$, and $b=2$.

The centre, (h, k), is at $(-1, 2)$. The major axis is parallel to the *y*-axis.



(2 marks)

The domain is $-3 \le x \le 1$, and the range is $-1 \le y \le 5$. **(2 marks)**

Section 2: Hyperbolas and Parabolas

Key Concepts

- hyperbola
- parabola
- vertex
- axis of symmetry
- asymptote
- standard form
- general form

The basic goals of this section are to ensure that students are able to

- classify and analyse hyperbolas and parabolas from their equations using a transformational approach
- convert the equations of hyperbolas and parabolas from general form to standard form and vice versa

Section 2: Assignment Answer Key (40 marks)

1. **a.** Because $\frac{y^2}{16}-\frac{x^2}{4}=1$ is the same as $\left(\frac{y}{4}\right)^2-\left(\frac{x}{2}\right)^2=1$, the graph of $y^2-x^2=1$ has been stretched horizontally by a factor of 2 and vertically by a factor of 4. **(2 marks)**

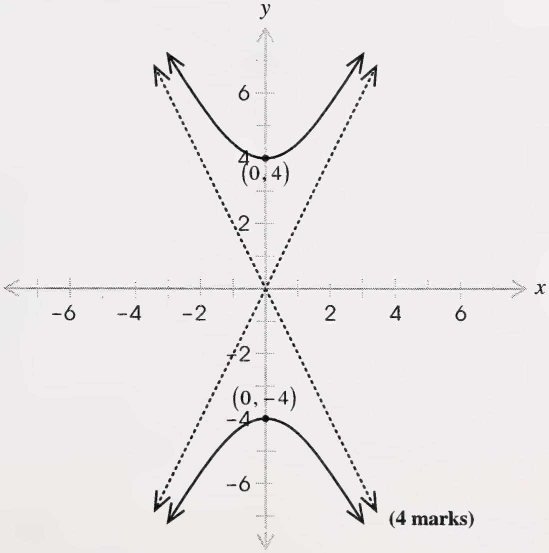
b. The slopes, m, of the asymptotes of $\frac{y^2}{16}-\frac{x^2}{4}=1$ are

$$\begin{aligned} m &= \pm\frac{1\times\text{vertical stretch}}{1\times\text{horizontal stretch}} \\ &= \pm\frac{4}{2} \\ &= \pm 2 \quad \textbf{(2 marks)} \end{aligned}$$

c. Compare $\frac{y^2}{16}-\frac{x^2}{4}=1$ with $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

$\therefore\ a=\sqrt{16}=4$ and $b=\sqrt{4}=2$

The vertices lie on the y-axis, $a=4$ units above and below the origin.



(4 marks)

2. Use $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Because the vertices are $(\pm 2, 0)$, $a=2$.

$$\begin{aligned}\therefore\quad \frac{x^2}{2^2}-\frac{y^2}{b^2}&=1\\ \frac{x^2}{4}-\frac{y^2}{b^2}&=1\end{aligned}$$

Because the curve passes through $(4, 3)$, substitute $x=4$ and $y=3$.

$$\begin{aligned}\frac{4^2}{4}-\frac{3^2}{b^2}&=1\\ 4-\frac{9}{b^2}&=1\\ 4-1&=\frac{9}{b^2}\\ 3&=\frac{9}{b^2}\\ b^2&=\frac{9}{3}\\ b^2&=3\end{aligned}$$

Therefore, the equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{3}=1$. **(3 marks)**

3. **a.**

$$\begin{aligned}(y-2)^2-\frac{x^2}{4}&=1\\ 4(y-2)^2-x^2&=4\\ 4\left(y^2-4y+4\right)-x^2&=4\\ 4y^2-16y+16-x^2&=4\\ 0&=x^2-4y^2+16y-12\\ x^2-4y^2+16y-12&=0\end{aligned}$$

(2 marks)

b. Compare $\frac{(y-2)^2}{1}-\frac{x^2}{4}=1$ with $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.

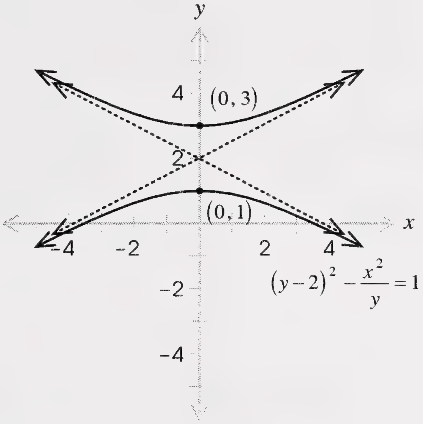
$\therefore h=0$, $k=2$, $a=\sqrt{1}=1$, and $b=\sqrt{4}=2$

The centre, (h, k), is at $(0, 2)$.

The hyperbola opens upward and downward; thus, the vertices are 1 unit above and below the centre.

The vertices are $(0, 2-1)=(0, 1)$ and $(0, 2+1)=(0, 3)$. **(2 marks)**

c. The slopes of the asymptotes $\pm\frac{a}{b}$ are $\pm\frac{1}{2}$. **(4 marks)**



4. a.

$$4x^2-y^2-8x-4y+16=0$$
$$4x^2-8x-y^2-4y=-16$$
$$4(x^2-2x)-(y^2+4y)=-16$$
$$4\left[(x^2-2x+1)-1\right]-\left[(y^2+4y+4)-4\right]=-16$$
$$4\left[(x-1)^2-1\right]-\left[(y+2)^2+4\right]=-16$$
$$4(x-1)^2-4-(y+2)^2+4=-16$$
$$4(x-1)^2-(y+2)^2=-16$$
$$\frac{4(x-1)^2}{-16}-\frac{(y+2)^2}{-16}=\frac{-16}{-16}$$
$$-\frac{4(x-1)^2}{16}+\frac{(y+2)^2}{16}=1$$
$$\frac{(y+2)^2}{16}-\frac{4(x-1)^2}{16}=1$$
$$\frac{(y+2)^2}{16}-\frac{(x-1)^2}{4}=1 \quad \textbf{(4 marks)}$$

b. Compare $\frac{(y+2)^2}{16}-\frac{(x-1)^2}{4}=1$ with $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.

$\therefore\ h=1,\ k=-2,\ a=\sqrt{16}=4$, and $b=\sqrt{4}=2$

The centre, (h, k), is at $(1, -2)$.

The hyperbola opens upward and downward. Thus, the vertices are $a=4$ units above and below the centre.

The vertices are $(1, -2-4)=(1, -6)$ and $(1, -2+4)=(1, 2)$. **(3 marks)**

c. The domain is the set of reals, and the range is $y\leq-6$ or $y\geq2$. **(2 marks)**

5. The reflection of $y=x^2$ in the x-axis is $y=-x^2$.

To stretch the graph of $y=-x^2$ vertically by a factor of 2, substitute $\frac{y}{2}$ for y into the equation.

$$\frac{y}{2}=-x^2$$
$$y=-2x^2$$

To translate this curve 3 units to the left and 1 unit down, substitute $(x+3)$ for x and $(y+1)$ for y.

$$y+1=-2(x+3)^2 \quad \longleftarrow \text{standard form}$$

Express the equation in general form.

$$y+1=-2(x^2+6x+9)$$
$$y+1=-2x^2-12x-18$$
$$2x^2+12x+y+19=0 \quad \longleftarrow \text{general form}$$

(5 marks)

6. a.
$$3y^2-x-12y+14=0$$
$$3y^2-12y=x-14$$
$$x-14=3y^2-12y$$
$$x-14=3(y^2-4y)$$
$$x-14=3\left[(y^2-4y+4)-4\right]$$
$$x-14=3\left[(y-2)^2-4\right]$$
$$x-14=3(y-2)^2-12$$
$$x-2=3(y-2)^2$$

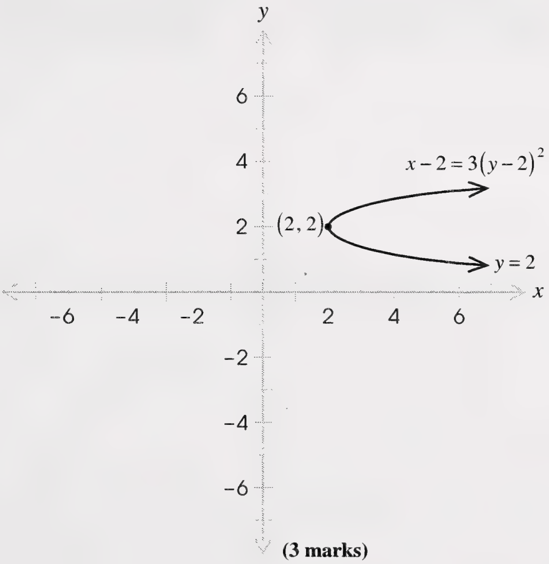
(4 marks)

b. Compare $x-2=3(y-2)^2$ with $x-h=a(y-k)^2$.

$\therefore h=2$, $k=2$, and $a=3$

The vertex, (h, k) is $(2, 2)$.

Because $a>0$, the graph opens to the right.

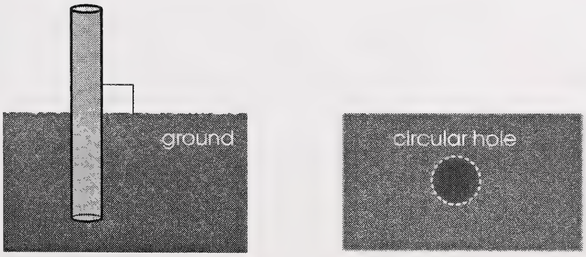


(3 marks)

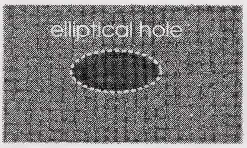
Final Module Assignment Answer Key (15 marks)

1. **a.** $C=2$ **(1 mark)**
   b. $C>0$ and $C\neq 2$ **(1 mark)**
   c. $C=0$ **(1 mark)**
   d. $C<0$ **(1 mark)**

2. If the peg was driven vertically into the ground, the hole is circular.



If the peg was driven into the ground at an angle other than 90°, the hole is elliptical at the surface.



(4 marks)

3. Use $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The centre of the ellipse is at $(2, 3)$; so, $h=2$ and $k=3$.

The major axis is 8 units long; so, $a=4$.

The minor axis is 4 units long; so, $b=2$.

$$\begin{aligned}
\therefore \quad \frac{(x-2)^2}{4^2}+\frac{(y-3)^2}{2^2}&=1\\
\frac{(x-2)^2}{16}+\frac{(y-3)^2}{4}&=1 \quad \longleftarrow \text{standard form}\\
(x-2)^2+4(y-3)^2&=16\\
x^2-4x+4+4\left(y^2-6y+9\right)&=16\\
x^2-4x+4+4y^2-24y+36&=16\\
x^2+4y^2-4x-24y+24&=0 \quad \longleftarrow \text{general form}
\end{aligned}$$

(5 marks)

4. To stretch $x^2+y^2-2x-3=0$ horizontally by a factor of 2, substitute $\frac{x}{2}$ for x.

$$\begin{aligned}
\left(\frac{x}{2}\right)^2+y^2-2\left(\frac{x}{2}\right)-3&=0\\
\frac{x^2}{4}+y^2-x-3&=0\\
x^2+4y^2-4x-12&=0 \quad \textbf{(2 marks)}
\end{aligned}$$

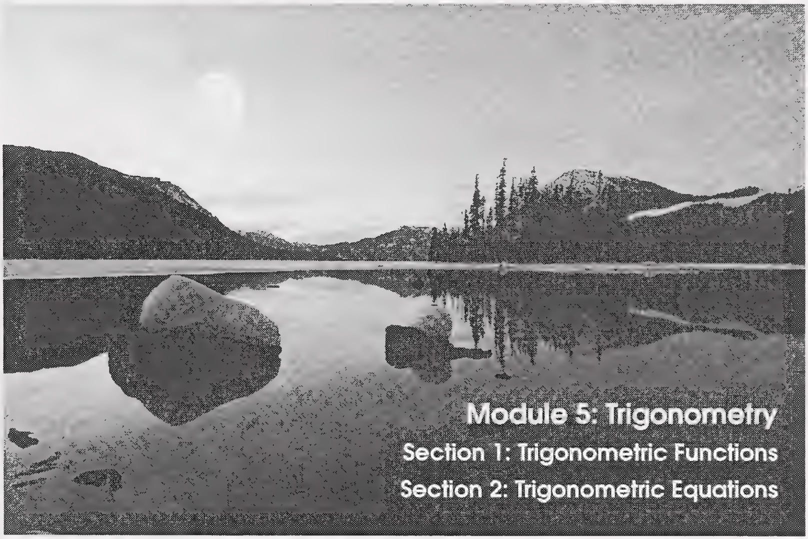
Module 5: Trigonometry

Overview

This module introduces students to trigonometry and trigonometric equations and identities.

In Section 1, students explore trigonometric functions. First, students explore the relationship between radians and degrees—the common angle measuring systems. Students then explore the definitions of the primary trigonometric functions—sine, cosine, and tangent—and their reciprocals—secant, cosecant, and cotangent—from the context of the unit circle. Students apply their knowledge of transformations to analyse the graphs of these functions. They apply this knowledge to model real-world situations, such as Ferris wheels, tides, and predicting temperatures.

In Section 2, students explore the solutions of equations containing trigonometric functions. First, they analyse the graphs arising from simple linear trigonometric equations. Then students use a graphing calculator as a tool in solving trigonometric equations and apply their knowledge of values of trigonometric functions to solve problems. In addition, students prove trigonometric identities and use the sum, difference, and double-angle identities to solve problems.



Module 5: Trigonometry
Section 1: Trigonometric Functions
Section 2: Trigonometric Equations

PHOTODISC, INC.

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 5A	
Section 1 Assignment	**30 marks**
Assignment Booklet 5B	
Section 2 Assignment	**35 marks**
Final Module Assignment	**35 marks**
TOTAL	**100 marks**

Section 1: Trigonometric Functions

Key Concepts

- angular measure
- trigonometric ratios
- sine, cosine, and tangent
- secant, cosecant, and cotangent
- graphs and transformations of sine and cosine
- applications of sinusoidal functions
- graphs and transformations of secant, cosecant, tangent, and cotangent

The basic goals of this section are to ensure that students are able to

- distinguish between degree and radian measure
- determine the exact and approximate values of trigonometric ratios for any multiples of 0°, 30°, 45°, 60°, and 90° and $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$
- describe the three primary trigonometric functions as circular functions with reference to the unit circle and an angle in standard position
- sketch and analyse the graphs of sine and cosine functions for
  - – amplitude
  - – period
  - – domain and range
  - – asymptotes, if any
  - – behaviour under transformations
- sketch and analyse the graphs of tangent, secant, cosecant, and cotangent functions for
  - – period
  - – domain and range
  - – asymptotes
  - – behaviour under transformations
- use trigonometric functions to model and solve problems

Section 1: Assignment Answer Key (30 marks)

1. C **(1 mark)**
2. B **(1 mark)**
3. C **(1 mark)**
4. B **(1 mark)**
5. A **(1 mark)**
6. D **(1 mark)**
7. C **(1 mark)**
8. B **(1 mark)**
9. A **(1 mark)**
10. D **(1 mark)**

11. **a.**

$$\begin{aligned}\theta &= \frac{\text{arc}}{\text{radius}}\\ &= \frac{11}{8}\\ &\doteq 1.4 \text{ rad}\end{aligned}$$

(1 mark)

b.

$$\begin{aligned}\theta &= \frac{\text{arc}}{\text{radius}}\\ 1.9 &= \frac{\text{arc}}{6.4}\\ \text{arc} &\doteq 12.2 \text{ rad}\end{aligned}$$

(1 mark)

12. Coterminal angles are found by adding multiples of 360°.

One positive coterminal angle is $-200° + 360° = 160°$. **(1 mark)**

One negative coterminal angle is $-200° - 360° = -560°$. **(1 mark)**

13. $$\begin{aligned}\frac{5\pi}{9}-\frac{-5\pi}{9}&=\frac{10\pi}{9}\\&\neq k(2\pi)\end{aligned}$$

Since the angles do not differ by a multiple of 2π, the angles are not coterminal. **(2 marks)**

14.

Degrees	45	211.99	228	−80.21	−36	105	34.5	−105.25
Radians	$\frac{\pi}{4}$	3.7	3.98	−1.4	$-\frac{\pi}{5}$	$\frac{7\pi}{12}$	0.60	$-\frac{9\pi}{16}$

(4 marks)

15. $$\begin{aligned}\sin\theta&=\frac{y}{\sqrt{x^2+y^2}}\\&=\frac{4}{\sqrt{3^2+4^2}}\\&=\frac{4}{5}\end{aligned}\qquad\begin{aligned}\cos\theta&=\frac{x}{\sqrt{x^2+y^2}}\\&=\frac{3}{\sqrt{3^2+4^2}}\\&=\frac{3}{5}\end{aligned}\qquad\begin{aligned}\tan\theta&=\frac{y}{x}\\&=\frac{4}{3}\\&=1\frac{1}{3}\end{aligned}$$ **(3 marks)**

16. The amplitude is $\sqrt{7}$.

The period is $\frac{360°}{5}=72°$.

The phase shift is 85° to the right.

The vertical displacement is 1.2 downward. **(3 marks)**

17. $$\begin{aligned}\text{Amplitude}&=\frac{|\max-\min|}{2}\\A&=\frac{|9-(-15)|}{2}\\&=\frac{|24|}{2}\\&=12\end{aligned}\qquad\begin{aligned}\text{Period}&=\frac{2\pi}{B}\\4&=\frac{2\pi}{B}\\B&=\frac{2\pi}{4}\\&=\frac{\pi}{2}\end{aligned}$$

For sine, the phase shift is 1 unit to the left; so, $C=1$. For cosine, there is no phase shift; so, $C=0$.

The midline of the graph occurs at $y=-3$; so, the vertical displacement is 3 units down. Thus, $D=-3$.

Therefore, the equations of the graph are $y=12\sin\left[\frac{\pi}{2}(x+1)\right]-3$ and $y=12\cos\left(\frac{\pi}{2}x\right)-3$. **(4 marks)**

Note to teacher: Sine and cosine equations may vary depending on the parameters selected.

Section 2: Trigonometric Equations

Key Concepts

- relating graphs and solutions
- solving trigonometric equations
- trigonometric identities
- sum, difference, Pythagorean, and double-angle identities

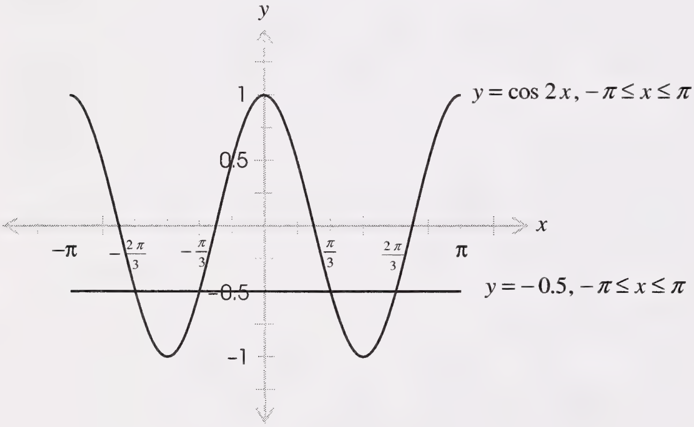
The basic goals of this section are to ensure that students are able to

- solve first- and second-degree trigonometric equations over a domain of length 2π algebraically and graphically
- determine the general solutions to trigonometric equations
- verify trigonometric identities numerically for any particular case and algebraically for general cases
- demonstrate trigonometric identities graphically
- use sum, difference, Pythagorean, and double-angle identities for sine and cosine to verify and simplify trigonometric expressions

Section 2: Assignment Answer Key (35 marks)

1. A **(1 mark)**
2. D **(1 mark)**
3. C **(1 mark)**
4. B **(1 mark)**
5. B **(1 mark)**
6. B **(1 mark)**
7. B **(1 mark)**

8.



Using special angles, $\cos x = -0.5$ when $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$ in its first period. Because $y = \cos 2x$ is a horizontal compression of $y = \cos x$ by a factor of 2, $\cos 2x = -0.5$ when $x = \frac{\frac{2\pi}{3}}{2}$ or $\frac{\pi}{3}$ and $x = \frac{\frac{4\pi}{3}}{2}$ or $\frac{2\pi}{3}$ in its first period, which is π.

Therefore, in the interval $-\pi \le x \le \pi$, the graphs of $y = \cos 2x$ and $y = -0.5$ intersect at $x = \pm\frac{\pi}{3}$ and $x = \pm\frac{2\pi}{3}$. **(4 marks)**

9. First, find a particular solution.

$$
\begin{aligned}
\sin\frac{x}{2} &= \frac{\sqrt{3}}{2}\\
\sin^{-1}\left(\sin\frac{x}{2}\right) &= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\\
\frac{x}{2} &= \frac{\pi}{3}\\
x &= \frac{2\pi}{3}
\end{aligned}
$$

Determine the period of $\sin\frac{x}{2}$.

$$
\begin{aligned}
\text{Period} &= \frac{2\pi}{|B|}\\
&= \frac{2\pi}{\frac{1}{2}}\\
&= 4\pi
\end{aligned}
$$

Therefore, the general solution to the equation $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ is $x = \frac{2\pi}{3} + 4k\pi$, where k is any integer. **(4 marks)**

10. **a.**

$$
\begin{aligned}
2\tan x + \sqrt{12} &= 0\\
2\tan x + 2\sqrt{3} &= 0\\
2\left(\tan x + \sqrt{3}\right) &= 0\\
\tan x + \sqrt{3} &= 0\\
\tan x &= -\sqrt{3}\\
x &= \frac{2\pi}{3} \text{ and } \frac{5\pi}{3}
\end{aligned}
$$

The solutions to the equation are $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$. **(2 marks)**

b.

$$
\begin{aligned}
2\cos^2 x + \cos x &= 1\\
2\cos^2 x + \cos x - 1 &= 0\\
(2\cos x - 1)(\cos x + 1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ 2\cos x - 1 &= 0 & \text{or} \quad \cos x + 1 &= 0\\
2\cos x &= 1 & \cos x &= -1\\
\cos x &= \frac{1}{2} & x &= \pi\\
x &= \frac{\pi}{3} \text{ and } \frac{5\pi}{3}
\end{aligned}
$$

The solutions to the equation are $x = \frac{\pi}{3}$, $x = \pi$, and $x = \frac{5\pi}{3}$. **(2 marks)**

11. Because the period of $2\tan x+\sqrt{12}$ is π, the general solution to $2\tan x+\sqrt{12}=0$ is $x=\frac{2\pi}{3}+k\pi$, where k is any integer. **(1 mark)**

Because the period of $\cos x$ is 2π, the general solutions to $2\cos^2 x+\cos x=1$ are $x=\frac{\pi}{3}+2k\pi,$ $x=\pi+2k\pi,$ and $x=\frac{5\pi}{3}+2k\pi,$ where k is any integer. **(1 mark)**

12. **a.**

LS	RS
$\sin\theta+\cos\theta\cot\theta$	$\csc\theta$
$=\sin 30°+\cos 30°\cot 30°$	$=\csc 30°$
$=\frac{1}{2}+\frac{\sqrt{3}}{2}\times\sqrt{3}$	$=2$
$=\frac{1}{2}+\frac{3}{2}$	
$=\frac{4}{2}$	
$=2$	

LS = RS

(2 marks)

b.

LS	RS
$\sin\theta+\cos\theta\cot\theta$	$\csc\theta$
$=\frac{\sin^2\theta}{\sin\theta}=\frac{\cos^2\theta}{\sin\theta}$	
$=\frac{1}{\sin\theta}$	
$=\csc\theta$	

LS = RS

(2 marks)

c. The relationship is true whenever both sides are defined. The only restriction is that $\sin\theta\neq 0$. So, the restriction to the equation is $x\neq k\pi$, where k is any integer. **(2 marks)**

13. **a.**

$$11\sin^2 y=13-3\sin^2 y$$
$$14\sin^2 y=13$$
$$\sin^2 y=\frac{13}{14}$$
$$\sin y=\pm\sqrt{\frac{13}{14}}$$
$$y\doteq 74.5°, 105.5°, 254.5°, \text{ and } 285.5°$$

Four angles satisfy this equation. **(2 marks)**

b. The second-quadrant angle that satisfies this equation is about 105.5°. **(1 mark)**

c. The fourth-quadrant angle that satisfies this equation is about 285.5°. **(1 mark)**

14.

$$2\sin^2\theta-7\sin\theta=-3$$
$$2\sin^2\theta-7\sin\theta+3=0$$
$$(\sin\theta-3)(2\sin\theta-1)=0$$

$\therefore\ \sin\theta-3=0$ or $2\sin\theta-1=0$

$\sin\theta=3$ $\qquad$ $\sin\theta=\frac{1}{2}$

$\theta=$ undefined $\qquad$ $\theta=30°$ and $150°$

In the interval $0<\theta\leq 360°$, $\theta=30°$ and $\theta=150°$. **(4 marks)**

Final Module Assignment Answer Key (35 marks)

1. D **(1 mark)**
2. B **(1 mark)**
3. D **(1 mark)**
4. A **(1 mark)**
5. B **(1 mark)**
6. B **(1 mark)**
7. D **(1 mark)**
8. B **(1 mark)**
9. C **(1 mark)**
10. B **(1 mark)**
11. A **(1 mark)**
12. C **(1 mark)**

13. Because the data starts with a maximum, use the cosine function, $y = A\cos B(x+C)+D$.

First, convert the maximum and minimum times to minutes of the day. Therefore, on December 21, the sun rises at $9(60)+17=557$; and on June 22, the sun rises at $4(60)+35=275$.

$$\begin{aligned}\text{Amplitude} &= \frac{|\text{max}-\text{min}|}{2}\\ A &= \frac{557-275}{2}\\ &= \frac{282}{2}\\ &= 141\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{2\pi}{B}\\ 365 &= \frac{2\pi}{B}\\ B &= \frac{2\pi}{365}\end{aligned}$$

Because the period starts on December 21 and not January 1, the phase shift is 10 days to the left. So, $C = 10$.

Because the midline of the curve occurs at $557-141=416$, the vertical displacement is 416 min up. So, $D = 416$.

Therefore, the equation that relates the time the sun rises in Prince Albert, Saskatchewan, to the day of the year is

$y = 141\cos\left[\frac{2\pi}{365}(x+10)\right]+416$. **(6 marks)**

14. a.

LS	RS
$\cos\theta + \cos\theta\tan^2\theta$	$\sec\theta$
$=\cos\theta(1+\tan^2\theta)$	
$=\cos\theta(\sec^2\theta)$	
$=\frac{1}{\sec\theta}(\sec^2\theta)$	
$=\sec\theta$	

LS = RS

(3 marks)

b.

LS	RS
$\cos\theta(\sec\theta-\csc\theta)$	$1-\cot\theta$
$=\cos\theta\sec\theta-\cos\theta\csc\theta$	
$=\frac{\cos\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}$	
$=1-\cot\theta$	

LS = RS

(3 marks)

15.

$$\sin 2y = \cos 4y$$
$$\sin 2y = \cos(2y + 2y)$$
$$\sin 2y = \cos 2y \cos 2y - \sin 2y \sin 2y$$
$$\sin 2y = \cos^2 2y - \sin^2 2y$$
$$\sin 2y = \left(1 - \sin^2 2y\right) - \sin^2 2y$$
$$\sin 2y = 1 - 2\sin^2 2y$$
$$2\sin^2 2y + \sin 2y - 1 = 0$$
$$(2\sin 2y - 1)(\sin 2y + 1) = 0$$

$\therefore\ 2\sin 2y - 1 = 0$ or $\sin 2y + 1 = 0$

$$2\sin 2y = 1 \qquad \sin 2y = -1$$
$$\sin 2y = \frac{1}{2} \qquad 2y = 270^\circ$$
$$2y = 30^\circ \text{ and } 150^\circ \qquad y = 135^\circ$$
$$y = 15^\circ \text{ and } 75^\circ$$

Because $\sin 2y$ has two periods in the interval $0^\circ \le y \le 360^\circ$, the solutions to the equation in the second period are $y = 15^\circ + 180^\circ$ or 195°, $y = 75^\circ + 180^\circ$ or 255°, and $y = 135^\circ + 180^\circ$ or 315°.

Therefore, the solutions to $\sin 2y = \cos 4y$ are $y = 15^\circ$, $y = 75^\circ$, $y = 135^\circ$, $y = 195^\circ$, $y = 255^\circ$, and $y = 315^\circ$. **(5 marks)**

16. $\cos 8^\circ \cos 38^\circ + \sin 8^\circ \sin 38^\circ = \cos(38 - 8)^\circ$

$$= \cos(30^\circ)$$
$$= \frac{\sqrt{3}}{2}$$ **(1 mark)**

17.

LS	RS
$\frac{\sin^2\theta}{1-\cos\theta}$	$\frac{\sec\theta+1}{\sec\theta}$
$=\frac{1-\cos^2\theta}{1-\cos\theta}$	$=\frac{\sec\theta}{\sec\theta}+\frac{1}{\sec\theta}$
$=\frac{(1+\cos\theta)\cancel{(1-\cos\theta)}}{\cancel{1-\cos\theta}}$	$=1+\frac{1}{\frac{1}{\cos\theta}}$
$=1+\cos\theta$	$=1+\cos\theta$

LS = RS

(5 marks)

Module 6: Combinatorics

Overview

This module introduces students to combinatorics—the branch of mathematics that deals with the study of permutations and combinations.

In Section 1, students start by examining tree diagrams and the Fundamental Counting Principle, then proceed to exploring permutations. Their exploration is restricted to linear permutations of sets or parts of sets. These linear permutations may involve like elements or problem situations with separate cases or imposed constraints. Lastly, students investigate combinations and problem situations involving combinations with or without constraints.

In Section 2, students use the knowledge of combinatorics, acquired in Section 1, to explore pathway problems, Pascal's triangle, and the Binomial Theorem. Pascal's triangle is the unifying theme in this section; students apply it to pathway problems and to expansions of powers of binomials.

Module 6: Combinatorics
Section 1: Permutations and Combinations
Section 2: Pathways, Pascal's Triangle, and the Binomial Theorem

EYEWIRE, INC

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 6A	
Section 1 Assignment	**40 marks**
Assignment Booklet 6B	
Section 2 Assignment	**35 marks**
Final Module Assignment	**25 marks**
TOTAL	**100 marks**

Section 1: Permutations and Combinations

Key Concepts

- tree diagram
- Fundamental Counting Principle
- permutations of a set of objects
- permutations of a subset of a set of objects
- distinguishable permutations
- combinations

The basic goals of this section are to ensure that students are able to

- use the Fundamental Counting Principle to determine the number of different ways to perform multi-step operations
- determine the number of permutations of *n* different objects taken *r* at a time and use this to solve problems
- determine the number of combinations of *n* different objects taken *r* at a time and use this to solve problems

Section 1: Assignment Answer Key (40 marks)

1. **a.** $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$ **(1 mark)**

b.

$$\begin{aligned} {}_7C_2 &= \frac{7!}{(7-2)!2!} \\ &= \frac{7!}{5!2!} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1)(2 \times 1)} \text{ or } \frac{7 \times 6}{2 \times 1} \text{ or } 7 \times 3 \quad \textbf{(1 mark)} \end{aligned}$$

c.

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \text{ or } 8 \times 7 \times 6 \quad \textbf{(1 mark)} \end{aligned}$$

2. **a.**

$$\begin{aligned} \frac{27 \times 26 \times 25 \times 24}{4 \times 3 \times 2 \times 1} &= \frac{27 \times 26 \times 25 \times 24 \times 23!}{4!23!} \quad \longleftarrow \text{Multiply the numerator and denominator by } 23!. \\ &= \frac{27!}{4!23!} \\ &= \frac{27!}{(27-23)!23!} \text{ or } \frac{27!}{(27-4)!4!} \\ &= {}_{27}C_{23} \text{ or } {}_{27}C_4 \quad \textbf{(1 mark)} \end{aligned}$$

b.

$$\begin{aligned} 30 \times 29 \times 28 \times 27 \times 26 &= \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25!}{25!} \\ &= \frac{30!}{(30-5)!} \\ &= {}_{30}P_5 \quad \textbf{(1 mark)} \end{aligned}$$

3. **a.** There are 30 students from which 10 must be chosen for the smaller group. Therefore, $n = 30$ and $r = 10$.

$$\begin{aligned}\therefore\ {}_nC_r &= \frac{n!}{(n-r)!r!}\\ {}_{30}C_{10} &= \frac{30!}{(30-10)!10!}\\ &= \frac{30!}{20!10!}\\ &= 30\ 045\ 015\end{aligned}$$

There are 30 045 015 different groups of ten students possible. **(2 marks)**

Note to teacher: Accept values of ${}_nC_r$ without intervening steps. These answers can be easily obtained using a graphing calculator.

b. Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of groups} &= {}_{30}C_{10} \times {}_6C_2\\ &= 30\ 045\ 015 \times 15\\ &= 450\ 675\ 225\end{aligned}$$

There are 450 675 225 different groups of ten students and two parents possible. **(2 marks)**

c. Only two parents are chosen for the smaller group. Since Kelly's mother is one of these two parents, the number of ways to choose the second parent is ${}_5C_1$.

Only ten students are chosen for the smaller group. Since Kelly will not be a part of this group, the number of ways of choosing the ten students is ${}_{29}C_{10}$.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of groups} &= 1 \times {}_5C_1 \times {}_{29}C_{10}\\ &= 1 \times 5 \times 20\ 030\ 010\\ &= 100\ 150\ 050\end{aligned}$$

There are 100 150 050 different groups of ten students and two parents possible. **(3 marks)**

4. **a.** Use the Fundamental Counting Principle. There are 6 ways to select each digit of the four-digit PIN.

$$\begin{aligned}\therefore\ \text{Number of PINs} &= 6 \times 6 \times 6 \times 6\\ &= 6^4\\ &= 1296\end{aligned}$$

There are 1296 different four-digit PINs possible. **(2 marks)**

b. Without repetition, you are arranging 4 numbers out of 6.

Therefore, $n = 6$ and $r = 4$.

$$\begin{aligned}\therefore {}_nP_r &= \frac{n!}{(n-r)!}\\ &= \frac{6!}{(6-4)!}\\ &= \frac{6!}{2!}\\ &= 360\end{aligned}$$

There are 360 different PINs possible. **(2 marks)**

c. Because Kerri does not want to use 6 of the possible PINs, she is left with $360 - 6 = 354$ PINs to choose from. **(2 marks)**

5. **a.** There is only one choice for the first digit. For the remaining digits, only 0 and 1 can be used. Therefore, there are only two choices for these digits.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of binary numbers} &= 1 \times 2 \times 2 \times 2 \times 2\\ &= 16\end{aligned}$$

There are 16 different binary numbers possible. **(1 mark)**

b. Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of binary numbers} &= 1 \times 2 \times 2 \times 1 \times 1\\ &= 4\end{aligned}$$

There are 4 different binary numbers that can be represented. **(1 mark)**

6. **a.** There are a total of $11 + 15 = 26$ students to choose from.

$\therefore n = 26$ and $r = 4$

$$\begin{aligned}\therefore {}_nC_r &= \frac{n!}{(n-r)!r!}\\ {}_{26}C_4 &= \frac{26!}{(26-4)!4!}\\ &= 14\ 950\end{aligned}$$

The school could form 14 950 different debate teams. **(2 marks)**

b. The number of ways of choosing two boys is ${}_{11}C_2$, and the number of ways of choosing two girls is ${}_{15}C_2$.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of teams} &= {}_{11}C_2 \times {}_{15}C_2\\ &= 55 \times 105\\ &= 5775\end{aligned}$$

The school could form 5775 different debate teams. **(2 marks)**

7. When 4 people are chosen from 12 people, the 12 people are divided into two groups: a group of 4 and a group of 8. Therefore, there are as many groups of 8 as there are groups of 4.

$\therefore\ {}_{12}C_4 = {}_{12}C_8$ **(2 marks)**

8. a. The order of the answers does matter.

$$\begin{aligned}\therefore\ \text{Number of ways} &= 4!\\ &= 24\end{aligned}$$

There are 24 ways to arrange the answers. **(1 mark)**

b. The letters A, B, C, and D can be thought of as A, C, and BD, where BD is one unit.

$$\begin{aligned}\therefore\ \text{Number of ways} &= 3!\\ &= 6\end{aligned}$$

The contestant could arrange the letters A, B, C, and D 6 different ways.

The possible arrangements are as follows:

A C B D
A B D C
B D A C
C A B D
C B D A
B D C A
(3 marks)

9. a. One character can be chosen in ${}_6C_1$, ways; one room can be chosen in ${}_9C_1$, ways; and one weapon can be chosen in ${}_6C_1$, ways.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of guesses} &= {}_6C_1 \times {}_9C_1 \times {}_6C_1\\ &= 6 \times 9 \times 6\\ &= 324\end{aligned}$$

There are 324 different guesses that can be made. **(2 marks)**

b. Since two characters have been eliminated, one character can be chosen in ${}_4C_1$ ways. Since four rooms have been eliminated, one room can be chosen in ${}_5C_1$ ways. As before, one weapon can be chosen in ${}_6C_1$ ways.

$$\begin{aligned}\therefore \text{ Number of guesses} &= {}_4C_1 \times {}_5C_1 \times {}_6C_1\\ &= 4\times5\times6\\ &= 120\end{aligned}$$

There are 120 different guesses that can be made. **(2 marks)**

10. The number of handshakes involving n people is ${}_nC_2$.

$$\begin{aligned}\therefore\ {}_nC_2 &= 190,\ n\geq 2\\ \frac{n!}{(n-2)!2!} &= 190\\ \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}2} &= 190\\ n(n-1) &= 380\\ n^2-n &= 380\\ n^2-n-380 &= 0\\ (n+19)(n-20) &= 0\end{aligned}$$

$$\begin{aligned}\therefore\ n+19&=0 \qquad\text{or}\qquad & n-20&=0\\ n&=-19 & n&=20\end{aligned}$$

Because $n\geq 2$, there were 20 people at the family reunion. **(3 marks)**

11. There are 7 letters in the word ARRANGE.
There are 2 Rs and 2 As.

$$\begin{aligned}\therefore \text{ Number of ways} &= \frac{7!}{2!2!}\\ &= 1260\end{aligned}$$

There are 1260 different ways to arrange the letters in the word ARRANGE. **(3 marks)**

Section 2: Pathways, Pascal's Triangle, and the Binomial Theorem

Key Concepts

- pathway
- Pascal's triangle
- Binomial Theorem
- general term formula

The basic goals of this section are to ensure that students are able to

- determine the number of pathways in a given compound pathway problem
- solve problems using the Binomial Theorem, limiting exponents to whole numbers

Section 2: Assignment Answer Key (35 marks)

1. The second term, ${}_{n}C_{1}$, is 10. Therefore, $n = 10$.

Determine the next two terms, ${}_{10}C_{3}$ and ${}_{10}C_{4}$, in the row.

$$\begin{aligned} {}_{n}C_{r} &= \frac{n!}{(n-r)!r!} \\ {}_{10}C_{3} &= \frac{10!}{(10-3)!3!} \\ &= \frac{10!}{7!3!} \\ &= 120 \end{aligned}$$

$$\begin{aligned} {}_{n}C_{r} &= \frac{n!}{(n-r)!r!} \\ {}_{10}C_{4} &= \frac{10!}{(10-4)!4!} \\ &= \frac{10!}{6!4!} \\ &= 210 \end{aligned}$$

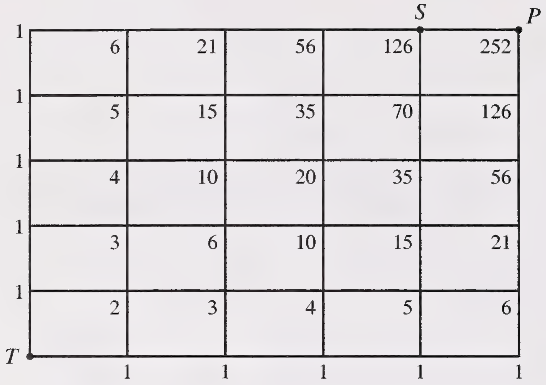
The next two terms are 120 and 210. **(3 marks)**

2. There are $n + 1$ terms in the expansion of $(a+b)^n$.

$$\begin{aligned} \therefore\ n + 1 &= 15 \\ n &= 14 \end{aligned}$$ **(1 mark)**

3. **a.** **Method 1: Using Pascal's Triangle**

Determine the number of paths to each vertex.



There are 252 different paths a person can take from the tennis courts to the swimming pool.

Method 2: Using Distinguishable Arrangements

In order for a person to walk from the tennis courts to the swimming pool, he or she must walk 5 blocks north and 5 blocks east.

$$\begin{aligned}\therefore \text{ Number of ways} &= \frac{10!}{5!5!}\\ &= 252\end{aligned}$$

There are 252 different paths a person can take from the tennis courts to the swimming pool. **(3 marks)**

b. According to the diagram in the answer to question 4.a., there are 126 different paths from the tennis courts to the swimming pool that pass through the store.

$$\begin{aligned}\therefore \text{ Percentage} &= \frac{126}{252} \times 100\%\\ &= 50\%\end{aligned}$$

Therefore, 50% of the paths pass through the store. **(3 marks)**

4. In this question, you must calculate ${}_8C_2 + {}_8C_3 + {}_8C_4 + {}_8C_5 + {}_8C_6 + {}_8C_7 + {}_8C_8$. Recall that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$.

$$\begin{aligned}\therefore \; {}_8C_0 + {}_8C_1 + {}_8C_2 + \cdots + {}_8C_8 &= 2^8\\ {}_8C_2 + {}_8C_3 + {}_8C_4 + \cdots {}_8C_8 &= 2^8 - {}_8C_0 - {}_8C_1\\ &= 256 - 1 - 8\\ &= 247\end{aligned}$$

There are 247 ways of selecting at least two people from eight people. **(3 marks)**

5. Each term in the expansion of $(x+y)^{11}$ is of the form ${}_{11}C_k x^{11-k} y^k$. The exponents of x and y sum to 11.

a. No, the exponent of the power of x is at most 11. **(1 mark)**

b. Yes, the exponents in $x^{10}y$ add up to 11. **(1 mark)**

c. No, the exponents in x^5y^5 do not add up to 11. **(1 mark)**

d. No, there will always be a variable (x or y or both) in this expansion. **(1 mark)**

6. **a.** $a = x, b = 2,$ and $n = 4$

$$\begin{aligned}(x+2)^4 &= {}_4C_0(x)^4(2)^0 + {}_4C_1(x)^3(2)^1 + {}_4C_2(x)^2(2)^2 + {}_4C_3(x)^1(2)^3 + {}_4C_4(x)^0(2)^4\\ &= (1)(x^4)(1) + 4(x^3)(2) + 6(x^2)(4) + 4(x)(8) + (1)(1)(16)\\ &= x^4 + 8x^3 + 24x^2 + 32x + 16 \quad \textbf{(2 marks)}\end{aligned}$$

b. $(2x-1)^5 = [2x+(-1)]^5$

$a = 2x,\ b = -1,$ and $n = 5$

$$\begin{aligned}\therefore\ (2x-1)^5 &= {}_5C_0(2x)^5(-1)^0 + {}_5C_1(2x)^4(-1)^1 + {}_5C_2(2x)^3(-1)^2 + {}_5C_3(2x)^2(-1)^3\\ &\quad + {}_5C_4(2x)^1(-1)^4 + {}_5C_5(2x)^0(-1)^5\\ &= 1(32x^5)(1) + 5(16x^4)(-1) + 10(8x^3)(1) + 10(4x^2)(-1) + 5(2x)(1) + 1(1)(-1)\\ &= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1 \quad \textbf{(3 marks)}\end{aligned}$$

c. $a = 1, b = 3y,$ and $n = 3$

$$\begin{aligned}(x+3y)^3 &= {}_3C_0(x)^3(3y)^0 + {}_3C_1(x)^2(3y)^1 + {}_3C_2(x)^1(3y)^2 + {}_3C_3(x)^0(3y)^3\\ &= 1(x^3)(1) + 3(x^2)(3y) + 3(x)(9y^2) + 1(1)(27y^3)\\ &= x^3 + 9x^2y + 27xy^2 + 27y^3 \quad \textbf{(3 marks)}\end{aligned}$$

7. The term in the expansion of $(x+3)^7$ with x^5 is ${}_7C_r(x)^5(3)^r$.

$$\begin{aligned}\therefore\ n - r &= 5\\ 7 - r &= 5\\ r &= 2\end{aligned}$$

$$\begin{aligned}\therefore\ {}_7C_r(x)^5(3)^r &= {}_7C_2(x)^5(3)^2\\ &= 21(x^5)(9)\\ &= 189x^5\end{aligned}$$

Therefore, $k = 189$. **(3 marks)**

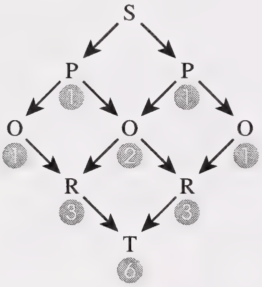
8. In the expansion of $(2x-3)^{10}$, there are $10+1=11$ terms. Therefore, the sixth term is the middle term.

$$\therefore\ a=2x,\ b=-3,\ n=10,\text{ and } r+1=6$$
$$r=5$$

$$t_{r+1}={}_nC_r a^{n-r}b^r$$
$$t_{5+1}={}_{10}C_5(2x)^{10-5}(-3)^5$$
$$t_6=252(2x)^5(-3)^5$$
$$=252(32x^5)(-243)$$
$$=-1\,959\,552x^5$$

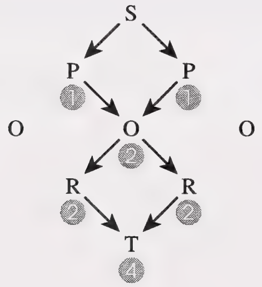
The coefficient is $-1\,959\,552$. **(3 marks)**

9. **a.** The number of paths from letter to letter is shown on the diagram.



There are 6 different paths that spell SPORT. **(2 marks)**

b.



There are 4 pathways that pass through the O in the middle. **(2 marks)**

Final Module Assignment Answer Key (25 marks)

1. **a.** For each cup, there are 2 choices: D or R.

Using the fundamental counting principle,

$$\begin{aligned}\text{Number of ways} &= 2\times 2\times 2\times 2\times 2\\ &= 32\end{aligned}$$

There are 32 ways the Ds and Rs can be recorded. **(2 marks)**

b. There are 3 Ds and 2 Rs.

$$\begin{aligned}\therefore\ \text{Number of arrangements} &= \frac{5!}{3!2!}\\ &= 10\end{aligned}$$

There are 10 possible ways the Ds and Rs can be recorded. **(2 marks)**

2. **a.** The order in which the books are chosen does not matter.

$\therefore\ n = 15$ and $r = 2$

$$\begin{aligned}\therefore\ {}_nC_r &= \frac{n!}{(n-r)!r!}\\ {}_{15}C_2 &= \frac{15!}{(15-2)!2!}\\ &= \frac{15!}{13!2!}\\ &= 105\end{aligned}$$

There were 105 different gifts possible. **(2 marks)**

b. The gift will consist of either 1 fiction book and 1 non-fiction book or 2 fiction books.

Case 1: 1 Fiction Book and 1 Non-Fiction Book

The number of ways to choose 1 fiction book is ${}_7C_1$.

The number of ways to choose 1 non-fiction book is ${}_8C_1$.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of gifts} &= {}_7C_1\times {}_8C_1\\ &= 7\times 8\\ &= 56\end{aligned}$$

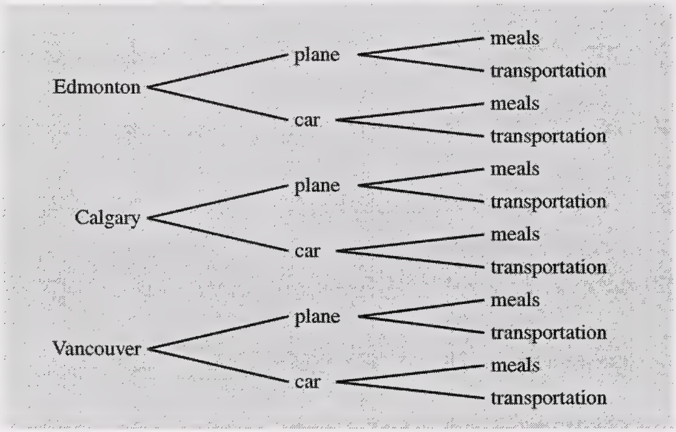
Case 2: 2 Fiction Books

The number of ways to choose 2 fiction books is ${}_7C_2$.

$$\begin{aligned}\therefore \text{ Number of gifts} &= {}_7C_2\\ &= \frac{7!}{(7-2)!2!}\\ &= \frac{7!}{5!2!}\\ &= 21\end{aligned}$$

Therefore, the total number of possible gifts is $21+56=77$. **(4 marks)**

3. The tree diagram is as follows:



(2 marks)

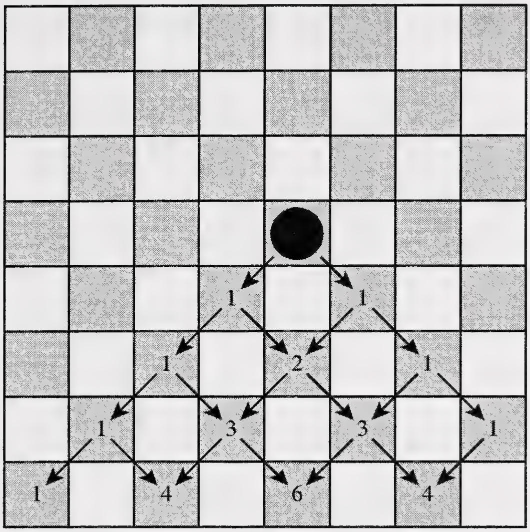
4. There are 3 possible CDs for the first case, 2 possible CDs for the second case, and 1 possible CD for the third case.

Using the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of ways} &= 3\times 2\times 1\\ &= 6\end{aligned}$$

There are 6 different ways the CDs can be put into the cases. **(2 marks)**

5. **a.**

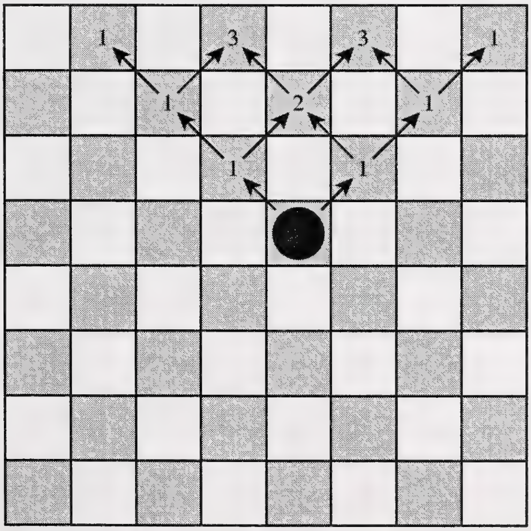


The total number of paths is the sum of the bottom row.

$1+4+6+4=15$

There are 15 different paths. **(3 marks)**

b. It is not possible to determine where the checker was initially placed because a path can lead from each starting place to the spot indicated on the board.



There are 8 paths that lead from the top of the board to the spot indicated on the board. **(3 marks)**

6. $(x-4)^5 = [x+(-4)]^5$

$\therefore\ a = x,\ b = -4,$ and $n = 5$

$$\begin{aligned}\therefore\ (x-4)^5 &= {}_5C_0(x)^5(-4)^0 + {}_5C_1(x)^4(-4)^1 + {}_5C_2(x)^3(-4)^2 + {}_5C_3(x)^2(-4)^3 + {}_5C_4(x)^1(-4)^4 + {}_5C_5(x)^0(-4)^5 \\ &= (1)(x^5)(1) + 5(x^4)(-4) + 10(x^3)(16) + 10(x^2)(-64) + 5(x)(256) + (1)(1)(-1024) \\ &= x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1024\end{aligned}$$

(3 marks)

7. The constant term in the expansion of $(x+2)^9$ will have x^0.

$$\begin{aligned}\therefore\ n - r &= 0 \\ 9 - r &= 0 \\ r &= 9\end{aligned}$$

Use the formula for the general term.

$$\begin{aligned}t_{r+1} &= {}_nC_r a^{n-r} b^r \\ t_{9+1} &= {}_9C_9(x)^{9-9}(2)^9 \\ t_{10} &= 1(x)^0(512) \\ &= 1(1)(512) \\ &= 512\end{aligned}$$

The constant term in the expansion of $(x+2)^9$ is 512. **(2 marks)**

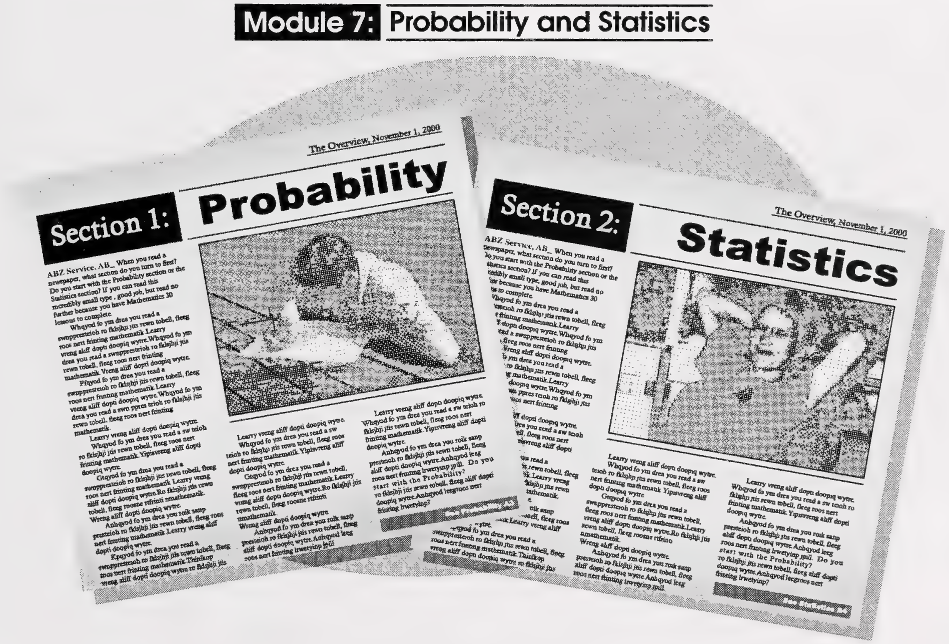
Module 7: Probability and Statistics

Overview

This module surveys probability theory and probability distributions as they apply in the analysis of statistics.

Section 1 introduces students to the fundamentals of probability—the terminology, definitions, and procedures. Students explore sample spaces, simple and compound events, and the probability theorems related to these events. These events include dependent and independent events, complementary events, and mutually exclusive and non-exclusive events. Students apply Venn and tree diagrams, the multiplication and addition theorems, and combinatorics to determine probabilities. Students also investigate procedures for conditional probability.

Section 2 introduces students to probability distributions and their applications to statistics. Students begin with an exploration of binomial distributions, then investigate the standard distribution as a generalization of the binomial distribution. Using technology, students determine means and standard deviations of sets of data. Students determine z-scores, use z-score tables and technology, and solve a variety of problems modelled by standard distributions. Students, then, are reintroduced to binomial distributions; and, when appropriate, they use the standard distribution approximation to explore these binomial distributions. In the last activity of this section, students use the normal distribution and the normal approximation to the binomial distribution to solve problems involving confidence intervals.


Module 7: Probability and Statistics
The Overview, November 1, 2000
Section 1: Probability
The Overview, November 1, 2000
Section 2: Statistics

Evaluation

The evaluation of this module will be based on three assignments:

Assignment Booklet 7A	
Section 1 Assignment	**35 marks**
Assignment Booklet 7B	
Section 2 Assignment	**35 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Section 1: Probability

Key Concepts

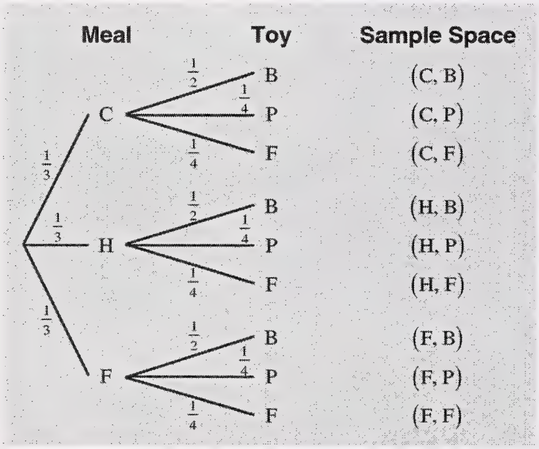
- sample space, event, and probability
- Venn diagram and tree diagram
- dependent and independent events
- complementary event
- mutually exclusive and non-exclusive events
- conditional probability and Bayes' Formula
- probabilities and combinatorics

The basic goals of this section are to ensure that students are able to

- construct a sample space for two or three events
- classify events as independent or dependent
- solve problems using the probabilities of mutually exclusive and complementary events
- determine the conditional probability of two events
- solve probability problems involving permutations, combinations, and conditional probability

Section 1: Assignment Answer Key (35 marks)

1.



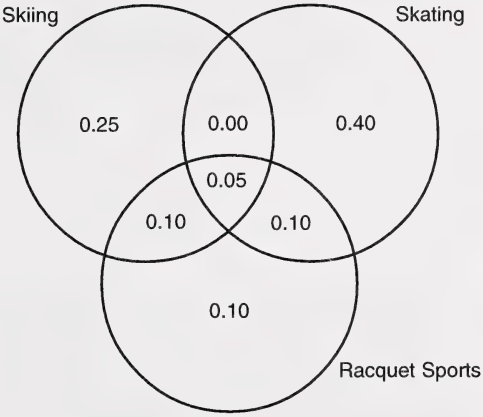
The probability that the meal is chicken or fish with a ball as the toy is

$$\begin{aligned}P(\text{C, B})+P(\text{F, B})&=\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}\\&=\frac{1}{6}+\frac{1}{6}\\&=\frac{2}{6}\\&=\frac{1}{3}\quad\textbf{(4 marks)}\end{aligned}$$

2. **a.** The two events are dependent because the taller the candle, the greater the mass. There is a relationship between mass and height. **(2 marks)**

   b. The two events are dependent because it is common to have blonde hair and blue eyes. **(2 marks)**

3. Use a Venn diagram to represent the data.



The probability that a person chosen at random only participates in skiing is 0.25. **(3 marks)**

4. **a.** The probability that the first two people have the same type of blood is

$$\begin{aligned}P &= P(\text{both have type O}) + P(\text{both have type A}) + P(\text{both have type B}) + P(\text{both have type AB})\\ &= (0.44)(0.44)+(0.42)(0.42)+(0.1)(0.1)+(0.04)(0.04)\\ &= 0.1936+0.1764+0.01+0.0016\\ &= 0.3816\end{aligned}$$

The probability that the first two people have the same type of blood is 0.3816 or 38.16%. **(3 marks)**

   b. The probability that a person does not have type AB blood is $1-0.04$ or 0.96.

The probability that the first five people don't have type AB blood is

$$\begin{aligned}P &= (0.96)(0.96)(0.96)(0.96)(0.96)\\ &\doteq 0.8154 \text{ or } 81.54\%\end{aligned}$$ **(3 marks)**

5. To determine the probability that at least two people roll the same number, it is easier to find the probability of the complementary event, which is nobody rolls the same number.

$$\begin{aligned}P(\text{no one rolls the same number}) &= 1\times\frac{5}{6}\times\frac{4}{6}\times\frac{3}{6}\\ &= \frac{{}_5P_3}{6^3}\\ &= \frac{60}{216}\\ &= \frac{5}{18}\end{aligned}$$

Therefore, the probability that at least two people roll the same number is

$$\begin{aligned} P &= 1-\frac{5}{18} \\ &= \frac{13}{18} \\ &\doteq 0.722 \text{ or } 72.2\% \end{aligned}$$ **(3 marks)**

6. **a.** $$\begin{aligned} P(\text{all show profit increase}) &= P(\text{Company A shows profit increase}) \times P(\text{Company B shows profit increase}) \\ &\quad \times P(\text{Company C shows profit increase}) \\ &= (0.70)(0.50)(0.65) \\ &= 0.2275 \end{aligned}$$

The probability that all three companies show a profit increase is 22.75%. **(2 marks)**

b. $$\begin{aligned} P(\text{Company A does not show profit increase}) &= 1-0.7 \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} P(\text{Company B does not show profit increase}) &= 1-0.5 \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} P(\text{Company C does not show profit increase}) &= 1-0.65 \\ &= 0.35 \end{aligned}$$

$$\begin{aligned} \therefore\ P(\text{at least one company shows profit increase}) &= 1-P(\text{not one company shows profit increase}) \\ &= 1-(0.3)(0.5)(0.35) \\ &= 1-0.0525 \\ &= 0.9475 \end{aligned}$$

The probability that at least one company shows a profit increase is 94.75%. **(2 marks)**

7. There is a total of $10+5+4=19$ people from which to choose a committee of 7 people. If the committee is comprised of all the adults that have no children, only $7-4=3$ other people need to be chosen.

The number of committees possible with the 4 adults without children is ${}_4C_4 \times {}_{15}C_3$.

The total number of committees is ${}_{19}C_7$.

$$\begin{aligned} \therefore\ P(\text{committee contains the 4 adults with no children}) &= \frac{\text{number of committees possible}}{\text{number of possible committees}} \\ &= \frac{{}_4C_4 \times {}_{15}C_3}{{}_{19}C_7} \\ &= 1\times\frac{455}{50\ 388} \\ &\doteq 0.009 \end{aligned}$$

The probability that the committee has 4 adults without children is approximately 0.9%. **(4 marks)**

8. $P(\text{used Internet or e-mail}) = P(\text{used Internet}) + P(\text{used e-mail}) - P(\text{used e-mail and Internet})$

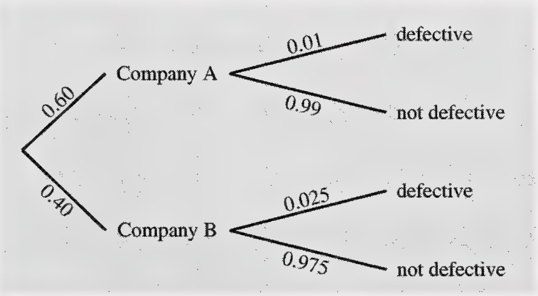
$$0.90 = 0.80 + 0.60 - P(\text{used e-mail and Internet})$$

$$\begin{aligned} P(\text{used e-mail and Internet}) &= 1.40 - 0.90 \\ &= 0.50 \end{aligned}$$

$$\begin{aligned} \therefore\ P(\text{used e-mail} \mid \text{used Internet}) &= \frac{P(\text{used e-mail and Internet})}{P(\text{used Internet})} \\ &= \frac{0.5}{0.8} \\ &= 0.625 \end{aligned}$$

The probability that the student used e-mail given the student used the Internet for projects is 62.5%. **(3 marks)**

9. Draw a probability tree.



There are two branches that lead to defective oven lights.

$$\begin{aligned} P(\text{defective}) &= P(\text{Company A, defective}) + P(\text{Company B, defective}) \\ &= (0.60)(0.01) + (0.40)(0.025) \\ &= 0.006 + 0.01 \\ &= 0.016 \end{aligned}$$

$$\begin{aligned} \therefore\ P(\text{Company A}) &= \frac{P(\text{Company A, defective})}{P(\text{defective})} \\ &= \frac{0.006}{0.016} \\ &= 0.375 \end{aligned}$$

The probability that a defective light was supplied by Company A is 37.5%. **(4 marks)**

Section 2: Statistics

Key Concepts

- binomial experiment
- binomial or Bernoulli trials
- binomial probability distribution
- histogram
- mean and standard deviation
- normal distribution
- z-scores and the standard normal distribution
- percentages and percentiles
- the normal distribution approximation of the binomial distribution
- confidence interval

The basic goals of this section are to ensure that students are able to

- solve probability problems using the binomial distribution
- find the population standard deviation of a data set or a probability distribution using technology
- use z-scores and z-score tables to solve problems
- use the normal distribution and the normal approximation to the binomial distribution to solve problems involving confidence intervals for large samples

Section 2: Assignment Answer Key (35 marks)

1. **a.** These are independent trials. The probabilities are modelled by the binomial distribution.

$n = 22$, $p = 0.76$, and $q = 0.24$

Since the probability is required for exactly half of the families, $x = 11$.

$$\begin{aligned}\therefore\ P &= {}_nC_x p^x q^{n-x}\\ &= {}_{22}C_{11}(0.76)^{11}(0.24)^{11}\\ &\doteq 0.005\end{aligned}$$

The probability that exactly half the students have families where the parents are married couples is about 0.5%. **(2 marks)**

b. "At least one student" means $x = 1, 2, 3, \ldots, 22$. The only event not considered here is the case when $x = 0$. It is easier to calculate the probability where $x = 0$ and subtract this from 1.

In this case a "success" is not married; therefore, $p = 0.24$.

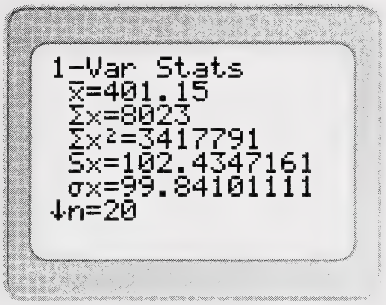
$$\begin{aligned}\therefore\ P(\text{at least } 1) &= 1 - P(0)\\ &= 1 - {}_{22}C_0(0.24)^{22}(0.76)^0\\ &\doteq 1\end{aligned}$$

The probability is approximately 1. There will be at least one student who does not live with parents that are married. **(2 marks)**

2. The mean will change unless the mean after 8 tests was 50%, since the average of 0 and 100% is 50%.

The standard deviation will increase because standard deviation is a measure of the variability of the data, and 0% and 100% lie at the extremes. **(2 marks)**

3.



```
1-Var Stats
 x̄=401.15
 Σx=8023
 Σx²=3417791
 Sx=102.4347161
 σx=99.84101111
↓n=20
```

$\therefore\ \bar{x} = 401.15$ mg and $\sigma \doteq 99.84$ mg **(2 marks)**

4. The height of the bar for "2 red" is 6.

Therefore, the probability that 2 red marbles are drawn is $\frac{6}{12}$ or 50%. **(1 mark)**

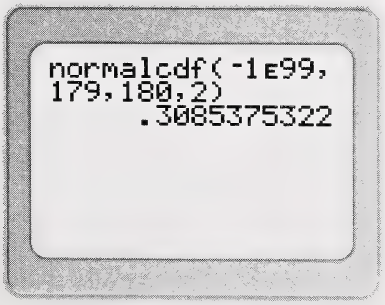
5. **a.**

$$\begin{aligned} z &= \frac{x-\bar{x}}{\sigma} \\ -1.5 &= \frac{x-180}{2} \\ x &= (-1.5)(2)+180 \\ &= 177 \end{aligned}$$

The box of cookies contained 177 g. **(2 marks)**

b. **Method 1: Using a Graphing Calculator**

$a = -1\text{E}99$, $b = 179$, $\bar{x} = 180$, and $\sigma = 2$



```
normalcdf(⁻1E99,
179,180,2)
        .3085375322
```

Method 2: Using Tables

Determine the z-score for 179.

$$\begin{aligned} z &= \frac{x-\bar{x}}{2} \\ &= \frac{179-180}{2} \\ &= -0.5 \end{aligned}$$

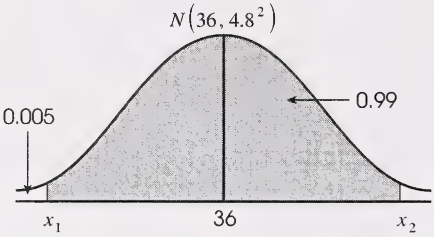
Using the tables, $P(z < -0.5) = 0.3085$.

The probability that a box of cookies has a mass less than 179 g is approximately 30.9%. **(3 marks)**

6. **a.** This is a binomial distribution, where $n = 100$, $p = 0.36$, and $q = 0.64$. Here a “success” is that a household uses the Internet.

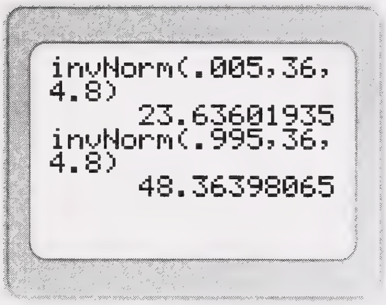
$$\therefore \bar{x} = np = 100(0.36) = 36 \qquad \sigma = \sqrt{npq} = \sqrt{100(0.36)(0.64)} = 4.8$$

Sketch the distribution.



Method 1: Using a Graphing Calculator

$\text{area}_1 = 0.005$, $\bar{x} = 36$, $\sigma = 4.8$, and $\text{area}_2 = 0.995$



```
invNorm(.005,36,
4.8)
        23.63601935
invNorm(.995,36,
4.8)
        48.36398065
```

The 99% confidence interval is approximately 23.6% to 48.4%.

Method 2: Using Tables

Using the tables, look for an area closest to 0.0050 and an area closest to 0.9950.

The corresponding z-score for an area of 0.0050 is -2.575 (the average of -2.57 and -2.58), and the corresponding z-score for an area of 0.9950 is 2.575 (the average of 2.57 and 2.58).

Use the z-scores to solve for x_1 and x_2.

$$\begin{aligned} z &= \frac{x-\bar{x}}{\sigma} \\ -2.575 &= \frac{x_1-36}{4.8} \\ x_1 &= (-2.575)(4.8)+36 \\ &= 23.64 \end{aligned}$$

$$\begin{aligned} z &= \frac{x-\bar{x}}{\sigma} \\ 2.575 &= \frac{x_2-36}{4.8} \\ x_2 &= (2.575)(4.8)+36 \\ &= 48.36 \end{aligned}$$

The 99% confidence interval is approximately 23.6% to 48.4%. **(3 marks)**

b. George found that 60% use the Internet. This does not fall in the 99% confidence interval. **(1 mark)**

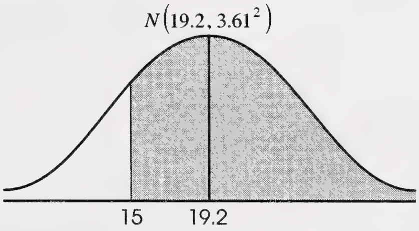
c. Answers will vary. George may live in an area where many people have home computers with Internet access. **(1 mark)**

7. a. This is a binomial distribution, where $n = 60$, $p = 0.32$, and $q = 0.68$.

$$\begin{aligned} \therefore\ \bar{x} &= np \\ &= 60(0.32) \\ &= 19.2 \end{aligned}$$

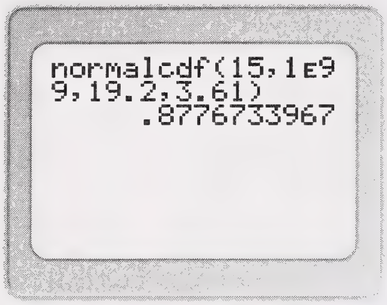
$$\begin{aligned} \sigma &= \sqrt{npq} \\ &= \sqrt{60(0.32)(0.68)} \\ &\doteq 3.61 \end{aligned}$$

Sketch the distribution.



Method 1: Using a Graphing Calculator

$a = 15$, $B = 1\text{E}99$, $\bar{x} = 19.2$, and $\sigma \doteq 3.61$



```
normalcdf(15,1E9
9,19.2,3.61)
        .8776733967
```

Method 2: Using Tables

Determine the z-scores for 15.

$$\begin{aligned} z &= \frac{x-\bar{x}}{\sigma} \\ &\doteq \frac{15-19.2}{3.61} \\ &\doteq -1.16 \end{aligned}$$

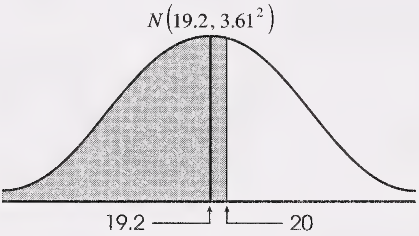
Using the tables, $P(z < -1.16) \doteq 0.1230$.

$$\begin{aligned} \therefore\ P(z > -1.16) &\doteq 1 - 0.1230 \\ &\doteq 0.877 \end{aligned}$$

The probability that more than 15 children are in day care at least 20 h per week is approximately 0.878 or 87.8%. **(4 marks)**

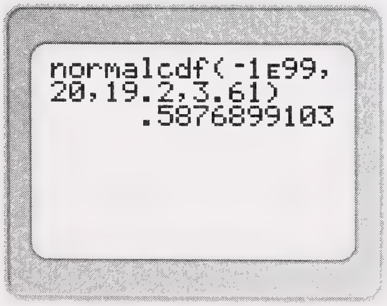
Note to teacher: Percentages differ slightly between Methods 1 and 2 due to rounding.

b. Sketch the distribution.



Method 1: Using a Graphing Calculator

$a = -1\text{E}99$, $b = 20$, $\bar{x} = 19.2$, $\sigma \doteq 3.61$



```
normalcdf(⁻1E99,
20,19.2,3.61)
        .5876899103
```

Method 2: Using Tables

Determine the z-score for 20.

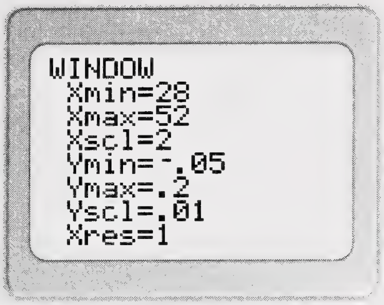
$$\begin{aligned} z &= \frac{x-\bar{x}}{\sigma} \\ &\doteq \frac{20-19.2}{3.61} \\ &\doteq 0.22 \end{aligned}$$

Using the tables, $P(z < 0.22) \doteq 0.5871$.

The probability that fewer than 20 children are in day care for at least 20 h per week is approximately 0.588 or 58.8%. **(4 marks)**

Note to teacher: Percentages differ slightly between Methods 1 and 2 due to rounding.

8. Answers may vary slightly. A sample answer is given.



```
WINDOW
 Xmin=28
 Xmax=52
 Xscl=2
 Ymin=-.05
 Ymax=.2
 Yscl=.01
 Xres=1
```

(2 marks)

9. This is a binomial distribution, where $n = 120$, $p = 0.05$, and $q = 0.95$. The value of x is 6, 7, or 8.

$$\begin{aligned}\therefore P(6,\ 7, \text{ or } 8 \text{ have lice}) &= P(6) + P(7) + P(8)\\ &= {}_{120}C_6\,(0.05)^6\,(0.95)^{114} + {}_{120}C_7\,(0.05)^7\,(0.95)^{113} + {}_{120}C_8\,(0.05)^8\,(0.95)^{112}\\ &\doteq 0.4110\end{aligned}$$

The probability that 6, 7, or 8 students will have head lice is approximately 41.1%. **(3 marks)**

10. $n = 500$, $p = 0.60$, and $q = 0.40$

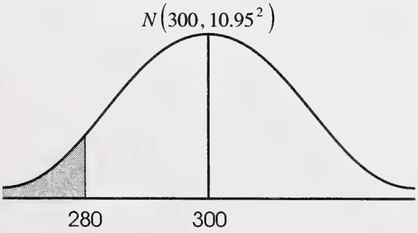
Because $np \geq 5$ and $nq \geq 5$, this is a binomial distribution that can be approximated by a normal distribution.

$$\begin{aligned}\therefore \bar{x} &= np & \sigma &= \sqrt{npq}\\ &= 500(0.60) & &= \sqrt{500(0.60)(0.40)}\\ &= 300 & &\doteq 10.95\end{aligned}$$

Determine the number of votes that make up 56% of the voters.

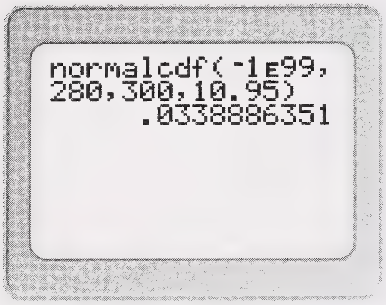
$$\begin{aligned}x &= 500 \times 0.56\\ &= 280\end{aligned}$$

Sketch the distribution.



Method 1: Using a Graphing Calculator

$a = -1\text{E}99, b = 280, \bar{x} = 300, \sigma \doteq 10.95$



```
normalcdf(⁻1E99,
280,300,10.95)
         .0338886351
```

Method 2: Using Tables

Determine the z-score for 280.

$$\begin{aligned} z &= \frac{x - \bar{x}}{\sigma} \\ &\doteq \frac{280 - 300}{10.95} \\ &\doteq -1.83 \end{aligned}$$

Using the tables, $P(z < -1.83) \doteq 0.0336$.

The probability that this candidate will get less than 56% of the votes is approximately 0.034 or 3.4%. **(3 marks)**

Final Module Assignment Answer Key (30 marks)

1. **a.** $P(\text{urban dweller and returns questionnaire}) = P(\text{urban dweller}) \times P(\text{returns the questionnaire})$

$$\begin{aligned} &= 0.77 \times 0.47 \\ &= 0.3619 \end{aligned}$$

The probability that a Canadian chosen is an urban dweller and returns the questionnaire is about 36.19%. **(2 marks)**

b. $P(\text{rural dweller and does not return questionnaire}) = P(\text{rural dweller}) \times P(\text{does not return the questionnaire})$

$$\begin{aligned} &= 0.23 \times 0.53 \\ &= 0.1219 \end{aligned}$$

The probability that a Canadian chosen is a rural dweller and does not return the questionnaire is about 12.19%. **(2 marks)**

2. Answers will vary. A sample answer is given.

The events are independent because it is unlikely that a young athlete could be spending time doing both sports at those levels. **(2 marks)**

3. There are 36 elements in the sample space of rolling two dice.

The following elements have two numbers that differ by 2:

$$\{(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)\}$$

The probability that the difference between the two numbers is 2 is $\frac{8}{36} = \frac{2}{9}$. **(3 marks)**

4. The number of ways the events can be arranged is $4!$.

$$\therefore P(\text{answers question correctly}) = \frac{\text{number of ways question can be answered correctly}}{\text{number of ways question can be answered}}$$

$$= \frac{1}{4!}$$

$$\doteq 0.0417$$

The probability of guessing correctly is approximately 4.2%. **(2 marks)**

5. a. The number of ways to select Tony is ${}_1C_1$.

The number of ways of selecting one of the remaining four students to accompany Tony is ${}_4C_1$.

Therefore, the number of ways of selecting Tony and one other student is ${}_1C_1 \times {}_4C_1$.

The number of ways of selecting any two students to attend the ceremony is ${}_5C_2$.

$$\therefore P(\text{Tony attends}) = \frac{{}_1C_1 \times {}_4C_1}{{}_5C_2}$$

$$= \frac{1 \times 4}{10}$$

$$= 0.4$$

The probability Tony will be chosen is 40%. **(3 marks)**

b. If Allecia has been chosen, only one additional person must be selected from the remaining four people. If Tony is among those four people, the probability that he will be chosen is $\frac{1}{4}$ or 25%. **(2 marks)**

6. a. The probability that a person has two bills to register with the same last three digits is

$$P = \frac{1}{10^3}$$

$$= \frac{1}{1000}$$ **(2 marks)**

b. It is easier to find the probability of the complementary event; that is, the probability that no two bills have the same last three digits.

$$P(10 \text{ bills with different last 3 digits}) = 1 \times \underbrace{\frac{999}{1000} \times \frac{998}{1000} \times \ldots \times \frac{991}{1000}}_{9 \text{ factors}}$$

$$= \frac{{}_{999}P_9}{1000^9}$$

$$\doteq 0.956$$

$$\therefore P(\text{at least 2 bills have the same last 3 numbers}) \doteq 1 - 0.956$$

$$\doteq 0.044$$

The probability that at least two bills have the same last three digits is approximately 4.4%. **(3 marks)**

7. The probability that the first VCR is not defective is $\frac{5}{9}$.

If the first VCR is not defective, the probability that the second VCR is not defective is $\frac{4}{9}$.

The probability that the third VCR is not defective is $\frac{3}{9}$, if the first two VCRs are not defective.

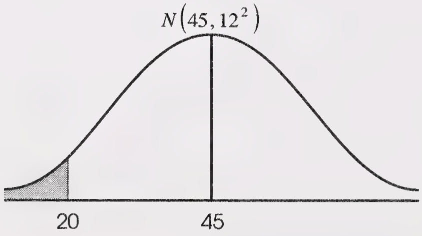
$$P(1, 2, \text{ or } 3 \text{ defects}) = 1 - P(0 \text{ defects})$$

$$= 1 - \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7}$$

$$\doteq 0.8810$$

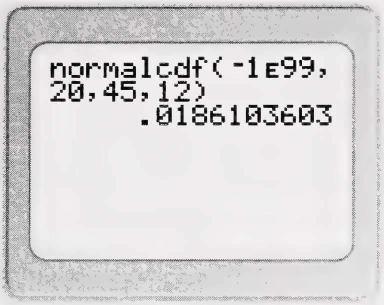
The probability that at least one of the first three VCRs sold is defective is approximately 88.1%. **(3 marks)**

8. **a.** Sketch the distribution.



Method 1: Using a Graphing Calculator

$a = -1\text{E}99, b = 20, \bar{x} = 45, \sigma = 12$



```
normalcdf(⁻1ᴇ99,
20,45,12)
        .0186103603
```

Method 2: Using Tables

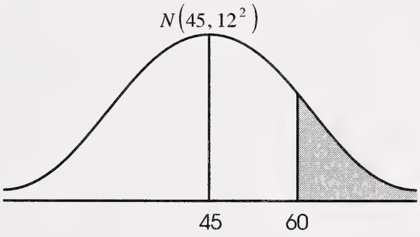
Determine the z-score for 20.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{20 - 45}{12} \doteq -2.08$$

Using the tables, $P(z < -2.08) \doteq 0.0188$

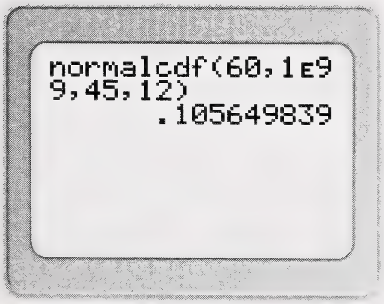
The probability of waiting less than 20 min is about 1.9%. **(2 marks)**

b. Sketch the distribution.



Method 1: Using a Graphing Calculator

$a = 60, b = 1\text{E}99, \bar{x} = 45$, and $\sigma = 12$



```
normalcdf(60,1ᴇ9
9,45,12)
         .105649839
```

Method 2: Using Tables

Determine the z-score for 60.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{60 - 45}{12} = 1.25$$

Using the tables, $P(z < 1.25) = 0.8944$.

$$\therefore\ P(z > 1.25) = 1 - 0.8944 = 0.1056$$

The probability of waiting more than 1 h is about 10.6%. **(2 marks)**

9. $p = 0.75$ and $q = 0.25$

The z-score that corresponds to a 90% (18 times out of 20) confidence interval is 1.645 (the average of 1.64 and 1.65).

For an error range of $\pm 2\%$,

$$1.645\sqrt{\frac{pq}{n}} = 0.02$$

$$1.645\sqrt{\frac{(0.75)(0.25)}{n}} = 0.02$$

$$\frac{1.645\sqrt{(0.75)(0.25)}}{\sqrt{n}} = 0.02$$

$$\frac{1.645\sqrt{(0.75)(0.25)}}{0.02} = \sqrt{n}$$

$$\frac{1.645^2(0.75)(0.25)}{0.02^2} = n$$

$$n = \frac{1.645^2(0.75)(0.25)}{0.02^2}$$

$$\doteq 1268.45$$

The sample size must have been about 1268 people. **(2 marks)**

Final Test

Security
Included here is the answer key to the Final Test and the student's copy of the Final Test. Teachers should keep these secure against unauthorized student access. Students should not have access to the test until it is assigned in a supervised situation. The answers should be stored securely and retained by the teacher at all times.

PURE MATHEMATICS 30

FINAL TEST ANSWER KEY

Part A: Multiple Choice and Numerical Response (65%)

Multiple Choice

1. A
2. C
3. B
4. A
5. D
6. C
7. B
8. D
9. B
10. D
11. C
12. A
13. B
14. A
15. C
16. C
17. B
18. B
19. A
20. D
21. C
22. C
23. B
24. D
25. A
26. C
27. B
28. C
29. D
30. D
31. A
32. B
33. C

Numerical Response

1.

0	.	6	7

2.

0	.	1	3

3.

5	0		

4.

4	.	0	

5.

3	6	0	

6.

0	.	0	8

Part B: Written Response (35%)

1. a.

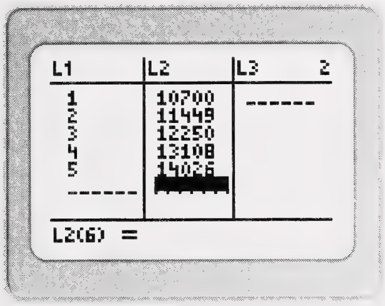
	A	B	C	D	E
1	Year	Opening Balance	Interest Rate	Interest	Closing Balance
2	1	$10,000.00	7.00%	$700.00	$10,700.00
3	2	$10,700.00	7.00%	$749.00	$11,449.00
4	3	$11,449.00	7.00%	$801.43	$12,250.43
5	4	$12,250.43	7.00%	$857.53	$13,107.96
6	5	$13,107.96	7.00%	$917.56	$14,025.52

(2 marks)

b. $a = 10\,000(1.07)$ and $r = 1.07$

$$\begin{aligned} \therefore t_n &= ar^{n-1} \\ &= 10\,000(1.07)(1.07)^{n-1} \\ &= 10\,000(1.07)^n \text{, where } n \in N \quad \textbf{(3 marks)} \end{aligned}$$

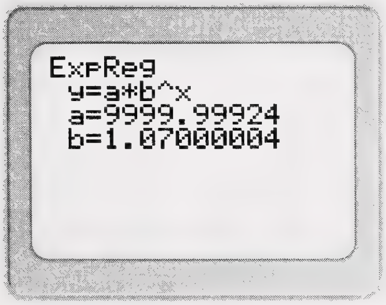
c. First, enter the year numbers $\{1, 2, 3, 4, 5\}$ in L1. Then enter the corresponding closing balances in L2.



```
L1      |L2      |L3      2
1       |10700   |------
2       |11449   |
3       |12250   |
4       |13108   |
5       |14026   |
------  |        |
L2(6) =
```

After entering the lists, press the following:

STAT [Select the CALC menu.] 0 (0:ExpReg) ENTER



```
ExpReg
 y=a*b^x
 a=9999.99924
 b=1.07000004
```

Rounding a to 10 000 and b to 1.07, the exponential function is $y = 10\,000(1.07)^x$. **(3 marks)**

d. The domain of the exponential function must correspond to the year numbers. Therefore, the domain is $x \in N$ or $x = 1, 2, 3, 4, 5, \ldots$. **(1 mark)**

e.

$$\begin{aligned} y &= 10\,000(1.07)^x \\ 20\,000 &= 10\,000(1.07)^x \\ 2 &= (1.07)^x \\ \log 2 &= \log(1.07)^x \\ \log 2 &= x \log 1.07 \\ x &= \frac{\log 2}{\log 1.07} \\ &\doteq 10.24 \end{aligned}$$

Because the number of years, x, must be a whole number, Leslie's investment would be at least $20 000 after 11 years. **(4 marks)**

f. No, the time required for the investment to double in value at 7% is independent of the original deposit.

If the original deposit is P, then $2P$ represents a doubling of the investment.

Substituting $2P$ for y and P for 10 000,

$$\begin{aligned} y &= 10\,000(1.07)^x \\ 2P &= P(1.07)^x \\ 2 &= (1.07)^x \end{aligned}$$

This is the same equation as in the answer to question 1.e. **(2 marks)**

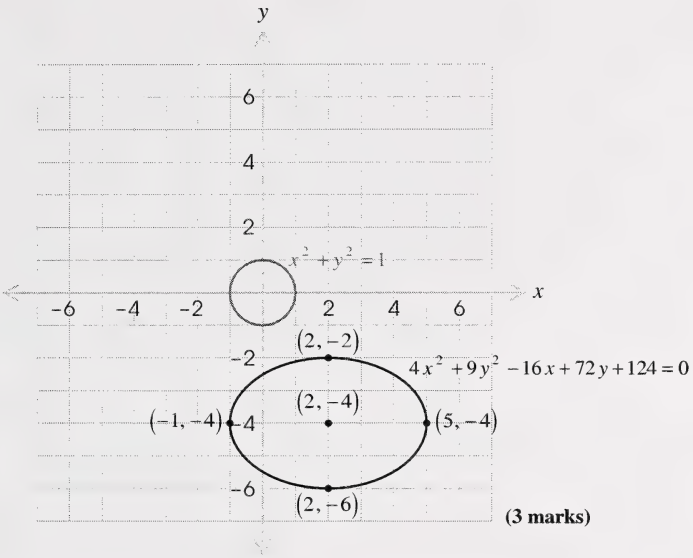
2. a. In this equation, $A = 4$ and $C = 9$. Therefore, because $AC > 0$ and $A \neq C$, this equation represents an ellipse. **(2 marks)**

b.

$$\begin{aligned} 4x^2 + 9y^2 - 16x + 72y + 124 &= 0 \\ 4x^2 - 16x + 9y^2 + 72y &= -124 \\ 4\left(x^2 - 4x\right) + 9\left(y^2 + 8y\right) &= -124 \\ 4\left[\left(x^2 - 4x + 4\right) - 4\right] + 9\left[\left(y^2 + 8y + 16\right) - 16\right] &= -124 \\ 4\left[(x-2)^2 - 4\right] + 9\left[(y+4)^2 - 16\right] &= -124 \\ 4(x-2)^2 - 16 + 9(y+4)^2 - 144 &= -124 \\ 4(x-2)^2 + 9(y+4)^2 &= 36 \\ \frac{\overset{1}{\cancel{4}}(x-2)^2}{\underset{9}{\cancel{36}}} + \frac{\overset{1}{\cancel{9}}(y+4)^2}{\underset{4}{\cancel{36}}} &= \frac{36}{36} \\ \frac{(x-2)^2}{9} + \frac{(y+4)^2}{4} &= 1 \end{aligned}$$

(3 marks)

c.

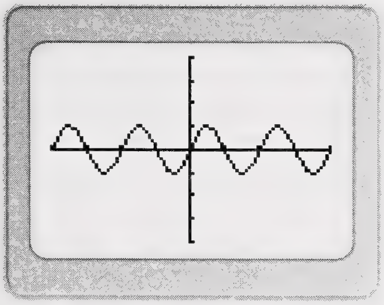


(3 marks)

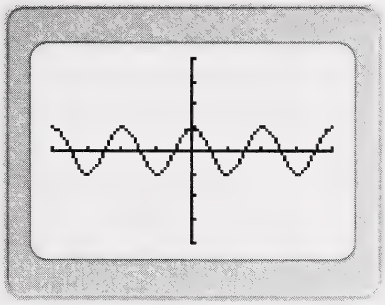
d. The ellipse is the unit circle stretched horizontally by a factor of 3 and vertically by a factor of 2, and then translated 2 units to the right and 4 units down. **(2 marks)**

3. a.

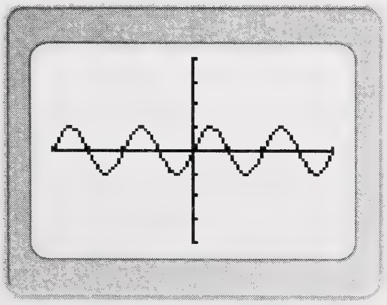
$$y=\frac{2\tan x}{1+\tan^2 x}$$



$y = \cos 2x$



$y = \sin 2x$



(4 marks)

b. Because the graphs of $y=\frac{2\tan x}{1+\tan^2 x}$ and $y=\sin 2x$ appear to be identical, the identity suggested by these graphs is $\frac{2\tan x}{1+\tan^2 x}=\sin 2x$.

(2 marks)

c.

LS	RS
$\frac{2\tan x}{1+\tan^2 x}$	$\sin 2x$ $=2\sin x\cos x$
$=\frac{2\frac{\sin x}{\cos x}}{\sec^2 x}$	
$=\frac{\frac{2\sin x}{\cos x}}{\frac{1}{\cos^2 x}}$	
$=\frac{2\sin x}{\cos x}\times\frac{\cos^2 x}{1}$	
$=2\sin x\cos x$	

LS = RS

$\therefore \frac{2\tan x}{1+\tan^2 x}=\sin 2x$ **(4 marks)**

PURE MATHEMATICS 30

FINAL TEST

GENERAL INSTRUCTIONS

This examination was developed to be completed in 2.5 h; however, you may take an additional 0.5 h to complete the examination.

This is a **closed-book** examination consisting of two parts:

PART A: Multiple Choice and Numerical Response

There are 33 multiple-choice questions and 6 numerical-response questions, of equal value, worth 65% of the examination.

PART B: Written Response

There are 3 written-response questions: 2 questions, of equal value, worth a total of 20% of the examination and 1 question worth 15% of the examination.

A tear-out formula sheet and *z*-score tables are included in this booklet.

You will require a graphing calculator. Approved graphing calculators are listed as follows:

- *Texas Instruments*: TI-82, TI-83, TI-83 Plus, TI-86, TI-89, TI-92, or TI-92 Plus
- *Casio*: CFX-9850GA-Plus, CFX-9850G, CFX-9800G, or FX-9700 series
- *Sharp*: EL-9600c, EL-9600, EL-9300, EL-9200

You may use any hand-held scientific calculator.

All calculator memories must be cleared before you begin the exam.

Value

PART A: MULTIPLE CHOICE AND NUMERICAL RESPONSE

65%

Multiple Choice

All multiple-choice questions must be answered on the Part A Response Page included in your test.

Read each question carefully and decide which of the choices BEST completes the statement or answers the question. Locate the question number on the Part A Response Page and place your answer in the corresponding blank.

Numerical Response

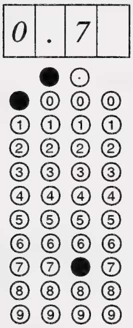
All numerical-response questions must be answered on the Part A Response Page included in your test.

For each question, record your answer by writing it in the boxes and then filling in the corresponding circles.

If an answer is a value between 0 and 1 (e.g., 0.7), be sure to record the 0 before the decimal place.

Enter the first digit of your answer in the left-hand box and leave any unused boxes blank.

Record 0.7 on the Part A Response Page. →



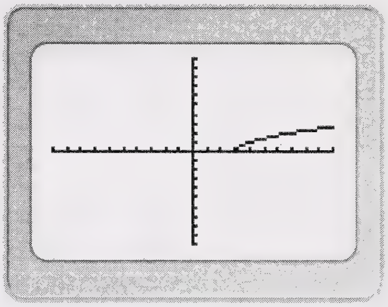
1. The graph of the function $y = f(3x)$ is the graph of $y = f(x)$.

A. compressed horizontally by a factor of 3
B. compressed vertically by a factor of 3
C. stretched horizontally by a factor of 3
D. stretched vertically by a factor of 3

Use the following information to answer question 2.

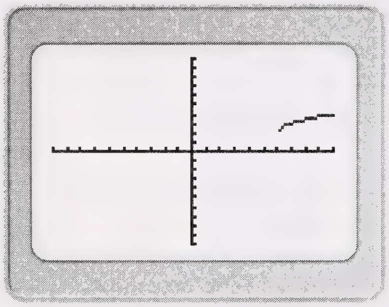
Using the standard window settings, Cheryl graphed $y = f(x)$ on her graphing calculator and obtained the graph on the right.

Using the same window settings, Cheryl then graphed $y - 2 = f(x + 3)$.

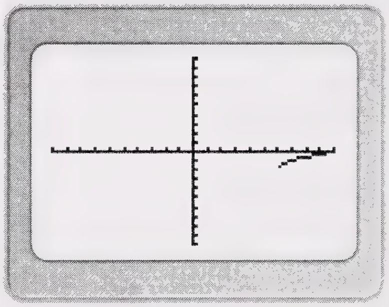


2. The display that best represents this function is

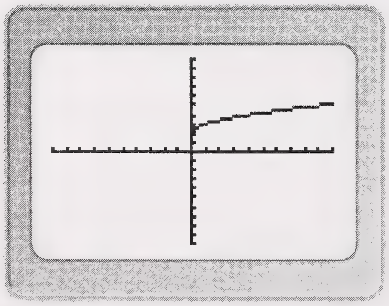
A.



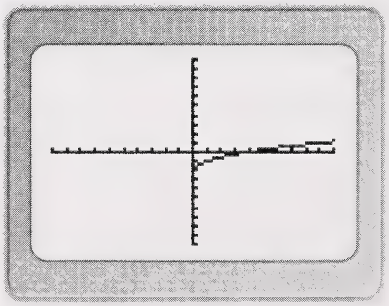
B.



C.



D.



3. To stretch the graph of $y = f(x)$ horizontally by a factor of 3 and then translate the resulting graph 2 units to the right, replace x in the equation by

A. $\frac{x+2}{3}$

B. $\frac{x-2}{3}$

C. $3(x+2)$

D. $3(x-2)$

4. If $f(x) = -3x + 2$, and $g(x) = \frac{1}{f^{-1}(x)}$, then $g(2)$ is

A. undefined

B. 3

C. 0

D. $-\frac{1}{3}$

5. If $y = f(x)$ is a quadratic function and the graph of $y = \frac{1}{f(x)}$ has two asymptotes, which statement must be true?

A. The graph of $y = f(x)$ has a positive y-intercept.

B. The graph of $y = f(x)$ lies entirely above the x-axis.

C. The graph of $y = f(x)$ is tangent to the x-axis.

D. The graph of $y = f(x)$ has two x-intercepts.

Numerical Response

1. If $2^{-x} = \left(\frac{1}{2}\right)^{4x-2}$, then the value of x, correct to the nearest hundredth, is ________.

(Record your answer in the numerical-response section of the Part A Response Page.)

6. If $3^x = 7$, then 3^{2x+1} is equal to

A. 42
B. 52
C. 147
D. 189

7. The graph of $y = \log_2 x - 0.5 \log_2 9$ is the same as the graph of

A. $y = \log_2 (3x)$

B. $y = \log_2 \frac{x}{3}$

C. $y = \log_2 (x - 3)$

D. $y = 3 \log_2 x$

8. If $\log_5 (3x - 4) + \log_5 (7x + 4) = \log_5 125$, then x is equal to

A. 25

B. $\frac{25}{2}$

C. 6

D. 3

9. What is the logarithmic form of $a^c = b$, where $a > 0, b > 0$, and $a \neq 1$?

A. $\log_b a = c$

B. $\log_a b = c$

C. $\log_a c = b$

D. $\log_c a = b$

10. The value of a rare stamp has doubled every 2 years. Its current value is $200. Assuming the same rate of growth, its value in 3 years, correct to the nearest dollar, will be

A. $500
B. $530
C. $550
D. $566

Numerical Response

2. The sum, S, of the terms of an infinite series is as follows:

$$S = 3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \ldots$$

The sum of the first n terms of this series is S_n.

The difference $S - S_4$, correct to the nearest hundredth, is ______.

(Record your answer in the numerical-response section of the Part A Response Page.)

11. In a geometric sequence, the third term is 12 and the sixth term is 96. The eleventh term of in this sequence is

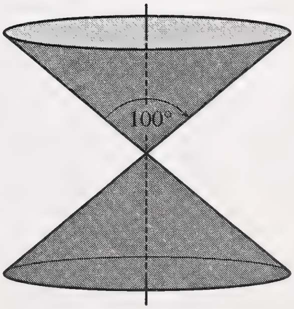
A. 768
B. 1536
C. 3072
D. 6144

12. The equation of a conic is described by $Ax^2 + Cy^2 + 2x - 4y - 8 = 0$, where $A \neq 0$ and $C \neq 0$. The two possible conic sections this equation could represent are a

A. a hyperbola and ellipse
B. a parabola and ellipse
C. a circle and parabola
D. a hyperbola and parabola

Numerical Response

3. The vertical angle of a double-napped, right circular cone is 100° (as shown). A plane cuts the cone. At what angle, relative to the axis of symmetry, should the plane cut the cone to produce a parabola? ______



(Record your answer in the numerical-response section of the Part A Response Page.)

13. The centre, *C*, and radius, *r*, of the circle $x^2+y^2-6x+8y=0$ are

A. $C(-3,-4)$ and $r=5$

B. $C(3,-4)$ and $r=5$

C. $C(-3,4)$ and $r=\sqrt{7}$

D. $C(3,-4)$ and $r=\sqrt{7}$

14. The unit circle, $x^2+y^2=1$, is stretched horizontally by a factor of 2, vertically by a factor of 3, and then translated 2 units to the left. The result is an ellipse. The equation in general form of this ellipse is

A. $9x^2+4y^2+36x=0$

B. $9x^2+4y^2-36x=0$

C. $4x^2+9y^2+16x-20=0$

D. $4x^2+9y^2-16x-20=0$

15. The slopes of the asymptotes of the hyperbola described by $4x^2-9y^2-72y-180=0$ are

A. $\pm\frac{4}{9}$

B. $\pm\frac{9}{4}$

C. $\pm\frac{2}{3}$

D. $\pm\frac{3}{2}$

16. The equation of the axis of symmetry of the parabola described by $x-2=-4(y+3)^2$ is

A. $x=-2$

B. $x=2$

C. $y=-3$

D. $y=3$

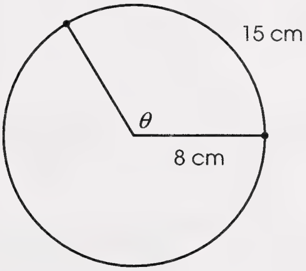
Numerical Response

4. The range of the trigonometric function $y = -3\sin 4x + d$ is $1 \leq y \leq 7$. The value of *d*, correct to the nearest tenth, is ________.

(Record your answer in the numerical-response section of the Part A Response Page.)

Use the following information to answer question 17.

In a circle with a radius of 8 cm, a central angle subtends an arc 15 cm long.



17. Correct to the nearest tenth of a degree, the measure of angle $\angle\theta$ is

A. 107.2°
B. 107.4°
C. 107.6°
D. 107.8°

18. Brandon used his graphing calculator, set in Radian mode, to determine all values of *x* for which $x - 3\sin 4x = 0$. The largest of these values, correct to the nearest tenth, is

A. 2.0
B. 2.2
C. 2.4
D. 2.6

19. The expression $\sin 2x + \sin 2x \tan^2 x$ is equivalent to

A. $2\tan x$

B. $2\cot x$

C. $2\sec x$

D. $2\csc x$

20. If $\cos x = \frac{1}{3}$, where $0 < x < \frac{\pi}{2}$, then the value of $\sin 2x$ is

A. $\frac{2}{3}$

B. $\frac{4}{9}$

C. $\frac{2\sqrt{2}}{3}$

D. $\frac{4\sqrt{2}}{9}$

21. The period of $y = 3\sin(4x - 5)$ is

A. $\frac{4\pi}{5}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{4}$

Numerical Response

5. The number of arrangements of all the letters of the word PERMUTE if the vowels are kept together is ______.

(Record your answer in the numerical-response section of the Part A Response Page.)

22. There are 7 distinct points on the circumference of a circle (as shown). The number of different quadrilaterals that can be formed using these points is

A. 210
B. 105
C. 35
D. 30

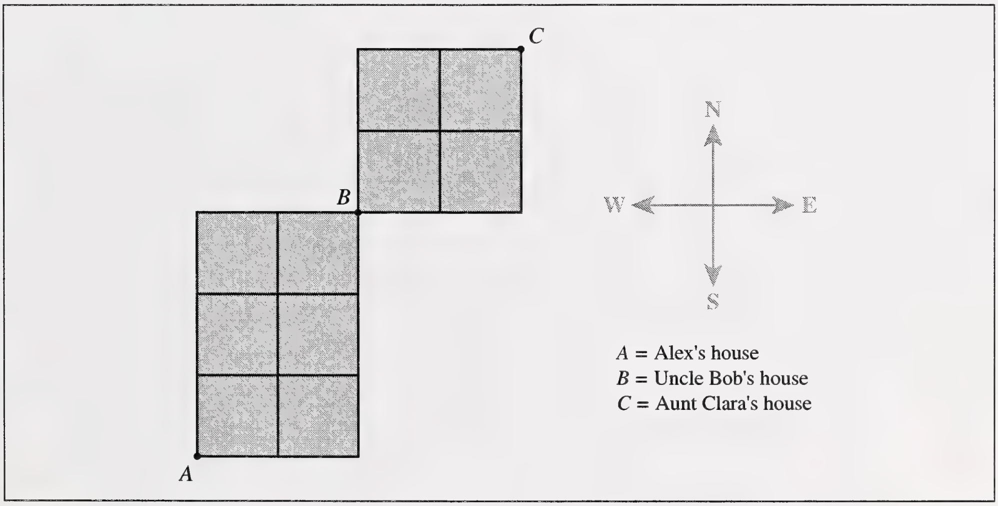
23. The number of different natural numbers greater than 2000 that can be formed using only 0, 1, 2, and 2 is

A. 3
B. 6
C. 12
D. 24

24. Anna collects hockey cards. She has cards of 30 different players: 5 goalies, 10 defencemen, and 15 forwards. The number of ways she can select 1 goalie, 2 defencemen, and 3 forwards, and then arrange these 6 cards in a row across the top of her dresser is

A. 3 071 250
B. 6 142 500
C. 36 855 000
D. 73 710 000

Use the following information to answer question 25.



A = Alex's house
B = Uncle Bob's house
C = Aunt Clara's house

25. Alex is walking from his house to his Aunt Clara's house, which is located 4 blocks east and 5 blocks north of his house. On the way, he plans to stop at his Uncle Bob's.

If Alex walks only north or east, the number of possible routes to Aunt Clara's house is

A. 60
B. 63
C. 120
D. 126

26. The coefficient of the middle term in the expression of $\left(4a-\frac{b}{2}\right)^{10}$ is

A. -252
B. 252
C. -8064
D. 8064

27. A bag contains 4 red marbles and 5 white marbles. Three marbles are removed from the bag without replacement. The probability that at least 1 of the marbles is red is

A. $\frac{5}{42}$
B. $\frac{37}{42}$
C. $\frac{1}{21}$
D. $\frac{20}{21}$

28. A coin is tossed and a die is thrown. The probability of a head on the coin or a 6 on the die is

A. $\frac{1}{12}$
B. $\frac{1}{3}$
C. $\frac{7}{12}$
D. $\frac{2}{3}$

29. A test consists of 6 multiple-choice questions with 5 choices for each question. Correct to the nearest hundredth, the probability of getting exactly 50% by guessing alone is

A. 0.32
B. 0.24
C. 0.16
D. 0.08

30. A jar contains 10 pennies and 10 dimes. Two coins are chosen at random without replacement. The probability of drawing a dime and a penny is

A. $\frac{1}{4}$

B. $\frac{5}{19}$

C. $\frac{1}{2}$

D. $\frac{10}{19}$

Numerical Response

6. A department store administers a test to new employees on store policy. The test consists of 10 multiple-choice questions with each having 4 choices. To pass the test, an employee must answer 5 questions correctly. Correct to the nearest hundredth, the probability of passing this test by guessing is ________.

(Record your answer in the numerical-response section of the Part A Response Page.)

31. A set of 1000 numbers has a mean of 200 and a standard deviation of 25. Then, 10 is added to each of the numbers in the set. The new set of numbers has

A. a mean of 210 and a standard deviation of 25
B. a mean of 210 and a standard deviation of 35
C. a mean of 200 and a standard deviation of 25
D. a mean of 200 and a standard deviation of 35

32. The raw scores of a test are normally distributed with a mean of 150 and a standard deviation of 10. Correct to the nearest tenth, the percentage of students who scored between 145 and 165 is

A. 80.2%
B. 62.5%
C. 46.7%
D. 43.3%

33. A sample of 250 city residents were asked about a proposed cat bylaw. According to this poll, 80% said they were in favour and 20% said they were opposed. The standard deviation, correct to the nearest tenth, of the respondents who were in favour is

A. 5.9
B. 6.1
C. 6.3
D. 6.5

PART A: RESPONSE PAGE

Multiple Choice

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______
7. ______
8. ______
9. ______
10. ______
11. ______
12. ______
13. ______
14. ______
15. ______
16. ______
17. ______
18. ______
19. ______
20. ______
21. ______
22. ______
23. ______
24. ______
25. ______
26. ______
27. ______
28. ______
29. ______
30. ______
31. ______
32. ______
33. ______

Numerical Response

1. ☐☐☐☐

⊙ ⊙
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① ① ① ①
② ② ② ②
③ ③ ③ ③
④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

2. ☐☐☐☐

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④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

3. ☐☐☐☐

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④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

4. ☐☐☐☐

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④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

5. ☐☐☐☐

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④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

6. ☐☐☐☐

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② ② ② ②
③ ③ ③ ③
④ ④ ④ ④
⑤ ⑤ ⑤ ⑤
⑥ ⑥ ⑥ ⑥
⑦ ⑦ ⑦ ⑦
⑧ ⑧ ⑧ ⑧
⑨ ⑨ ⑨ ⑨

Value

PART B: WRITTEN RESPONSE

35% Write your answers in the spaces provided as neatly as possible. For full marks, your answers must address all aspects of the question. Descriptions and/or explanations of concepts must be correct and include pertinent ideas, diagrams, calculations, and formulas.

Your answers must be presented in a well-organized manner using complete sentences and correct units.

Use the following information to answer question 1.

Leslie has had a part-time job during high school and has saved $10 000. She has just purchased a 5-year bond, compounded 7% per annum. Leslie is preparing a spreadsheet to determine the amount she will have at the end of each year.

	A	B	C	D	E
1	Year	Opening Balance	Interest Rate	Interest	Closing Balance
2	1	$10,000.00	7.00%	$700.00	$10,700.00
3	2	$10,700.00	7.00%	$749.00	$11,449.00
4	3	$11,449.00	7.00%	$801.43	$12,250.43
5	4				
6	5				

Written Response—15%

(2 marks) **1. a.** Complete Leslie’s spreadsheet for Years 4 and 5.

	A	B	C	D	E
1	Year	Opening Balance	Interest Rate	Interest	Closing Balance
2	1	$10,000.00	7.00%	$700.00	$10,700.00
3	2	$10,700.00	7.00%	$749.00	$11,449.00
4	3	$11,449.00	7.00%	$801.43	$12,250.43
5	4				
6	5				

(3 marks)

b. Before rounding, the closing balances (\$10 700, \$11 449.00, \$11 250.43, …) form a geometric sequence. Determine an expression for the general term of this sequence.

(3 marks)

c. You can also use the Exponential Regression feature on your graphing calculator to obtain an equation of the form $y = ab^x$ from which you can predict the closing balance for each year. Outline the steps you must follow to obtain this equation, and write your results. Round your answer where appropriate.

(1 mark) **d.** Explain how must you limit the domain of the function in question 1.c. so it correctly models the problem.

____

____

____

(4 marks) **e.** If Leslie can extend the term of this bond at the same interest rate, determine algebraically how many years it will be before the closing balance is at least \$20 000.

(2 marks) **f.** Does the time required for the investment to double in value at 7% depend on the size of the original deposit? Explain.

Use the following information to answer question 2.

A second-degree equation in two variables represents a conic section. Consider the following equation:

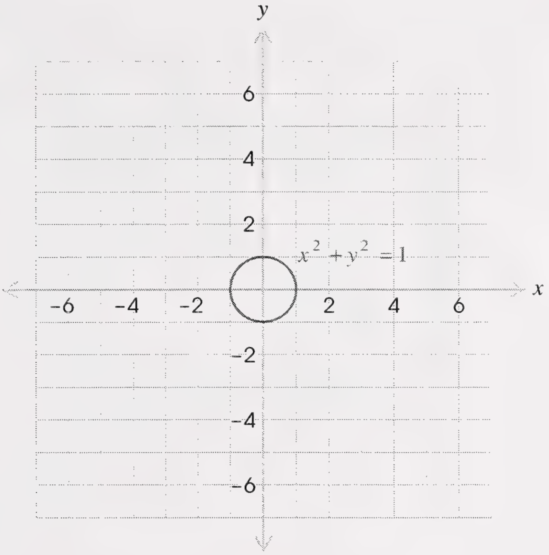
$$4x^2 + 9y^2 - 16x + 72y + 124 = 0$$

Written Response—10%

(2 marks) **2.** **a.** What kind of conic does this equation likely represent? Explain.

(3 marks) **b.** Write the equation in standard form.

(3 marks) **c.** Sketch the resulting conic. Clearly show the centre and the endpoints of the axes of symmetry.



(2 marks) **d.** Describe the transformations required to transform the graph of the unit circle, $x^2+y^2=1$, to the graph obtained in question 2.c.

____

____

____

____

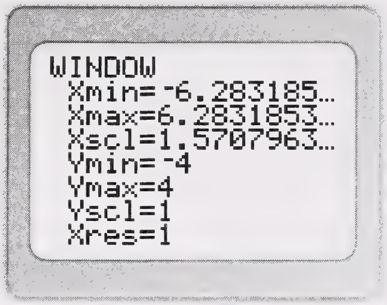
Use the following information to answer question 3.

Graphs of functions can be used to determine their similarities and differences. You can use a graphing calculator to obtain these graphs.

Written Response—10%

(4 marks) 3. **a.** Use your graphing calculator to graph each of the following functions.

Use the following window settings.



```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=1.5707963…
 Ymin=-4
 Ymax=4
 Yscl=1
 Xres=1
```

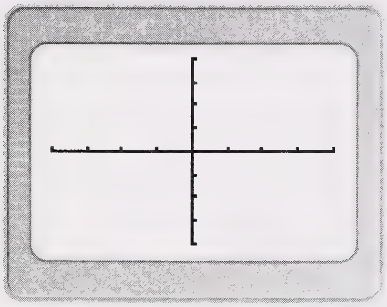
$\text{Xmin} = -2\pi$

$\text{Xmax} = 2\pi$

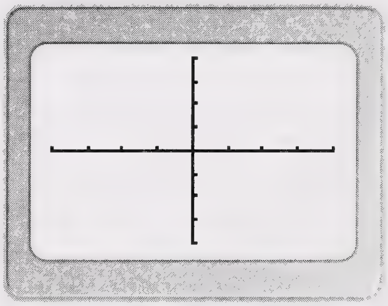
$\text{Xscl} = \frac{\pi}{2}$

Sketch each curve on the display given for each equation.

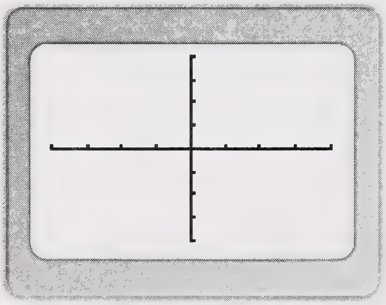
$$y = \frac{2\tan x}{1+\tan^2 x}$$



$$y = \cos 2x$$



$$y = \sin 2x$$



(2 marks) **b.** What trigonometric identity is suggested by these graphs? Explain.

(4 marks) **c.** Prove the identity in question 3.b. algebraically.

Pure Mathematics 30: Formula Sheet

For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Graphing Calculator Window Format

$$\left[x_{\text{min}}, x_{\text{max}}, x_{\text{scl}}\right]$$
$$\left[y_{\text{min}}, y_{\text{max}}, y_{\text{scl}}\right]$$

Exponents, Logarithms, and Geometric Series

$$\log_a (M \times N) = \log_a M + \log_a N$$

$$\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$$\log_a M^n = n \log_a M$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

$$t_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$$S_n = \frac{rt_n - a}{r - 1}, r \neq 1$$

$$S = \frac{a}{1 - r}, |r| < 1$$

Conics

General Form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Standard Form

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = \pm 1$$

$$y - k = a(x - h)^2$$

$$x - h = a(y - k)^2$$

Permutations and Combinations

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

In the expansion of $(x+y)^n$, the general term is

$t_{k+1} = {}_nC_k x^{n-k} y^k$.

Statistics

$\mu = np$

$\sigma = \sqrt{np(1-p)}$

$$z = \frac{x - \mu}{\sigma}$$

Probability

$$P(A \text{ and } B) = P(A) \times P(B/A)$$

$$P(k) = {}_nC_k\, p^k (1-p)^{n-k}$$

Trigonometry

$$a = r\theta$$

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

Areas Under the Standard Normal Curve, to the Left of *z*

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0880	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1432	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2626	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Areas Under the Standard Normal Curve, to the Left of *z*

	0.00	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6717
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7537	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9939	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

TEACHER QUESTIONNAIRE FOR PURE MATHEMATICS 30
(© 2000)

This course is designed in a new distance learning format, so we are interested in your responses. Your constructive comments will be greatly appreciated, as future course revisions can then incorporate any necessary improvements.

Teacher's Name ______________________ Area of Expertise ______________________

School Name ______________________ Date ______________________

Design

1. The modules follow a definite systematic design. Did you find it easy to follow?

☐ Yes ☐ No If no, explain.

____

____

2. Did your observations reveal that the students found the design easy to follow?

☐ Yes ☐ No If no, explain.

____

____

3. Did you find the Teacher's Guide helpful?

☐ Yes ☐ No If no, explain.

____

____

4. Part of the design involves stating the objectives in student terms. Did you find this helped the students understand what they were going to learn?

☐ Yes ☐ No If no, explain.

____

____

5. The Teacher's Guide contains answers to the questions in the Assignment Booklet and may include a sample test. Did you find these helpful?

☐ Yes ☐ No If no, explain.

____

____

6. Did the Follow-up Activities prove to be helpful?

☐ Yes ☐ No If no, explain.

____

____

7. Were students motivated to try these Follow-up Activities?

☐ Yes ☐ No If no, give details.

____

____

8. Suggestions for computer and video activities may be included in the course. Were your students able to use these activities?

☐ Yes ☐ No Comment on the lines below.

____

____

9. Were the assignments appropriate?

☐ Yes ☐ No If no, give details.

____

____

10. Did you fax assignments? ☐ Yes ☐ No

11. If you did fax, did you get satisfactory results from using this procedure?

☐ Yes ☐ No If no, give details.

____

____

Instruction

1. Did you find the instruction clear?

☐ Yes ☐ No If no, give details.

2. Did your observations reveal that the students found the instruction interesting?

☐ Yes ☐ No If no, give details.

3. Did you find the instruction adequate?

☐ Yes ☐ No If no, give details.

4. Was the reading level appropriate?

☐ Yes ☐ No If no, give details.

5. Was the workload adequate?

☐ Yes ☐ No If no, give details.

6. Was the content accurate and current?

☐ Yes ☐ No If no, give details.

7. Did the content flow consistently and logically?

☐ Yes ☐ No If no, give details.

8. Was the transition between booklets smooth?

☐ Yes ☐ No If no, give details.

9. Was the transition between print and media smooth?

☐ Yes ☐ No If no, give details.

Additional Comments

Thanks for taking the time to complete this questionnaire. Your feedback is important to us. Please return this questionnaire to the address on the right.

Fax Number: (780) 674-6561

Instructional Design and Development
Learning Technologies Branch
Box 4000
Barrhead, Alberta
T7N 1P4

Note: Please ensure that each of your students has completed and forwarded a copy of the Course Survey.

Pure Mathematics 30

Teacher's Guide

LRDC

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